Hypothesis Testing: Chi-Square Test[1]

November 9, 2017

[1] HMS, 2017, v1.0

Chapter References

- Diez: Chapter 6.3
- Navidi, Chapter 6.10

Chi-square Distributions

- Let $X_1, X_2, \ldots X_n$ be independent normally distributed random variables.
- Form the sum of their squares:

$$Q = \sum_{i=1}^{k} X_i^2$$

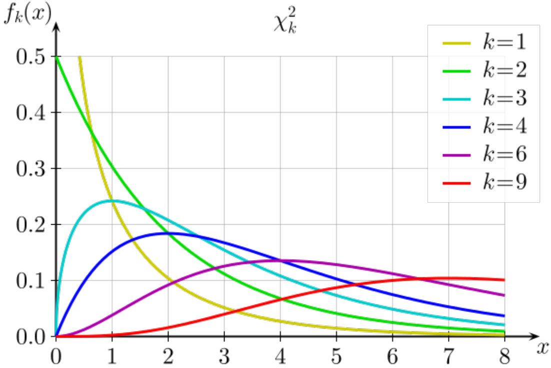
- Q is then distributed according to the χ^2 distribution.
- The chi-square distribution has one parameter, the degrees of freedom, k

$$f(x;\, k) = \frac{x^{(k/2-1)} e^{-x/2}}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} \qquad x > 0$$

Γ is the gamma function which generalizes the factorial to non-integer values. For every value for the degrees of freedom there is a unique curve described by $f(x;\, k)$.

Mean $= \mu = n$ variance $\sigma^2 = 2n$.

Chi-square Distributions



Chi-square Distributions: Its Purpose

- The purpose of χ^2 is to compare a set of data against a specific distribution.
- For example, the likelihood of throwing a particular number on a die is $1/6$
- What if we had a die that was suspect?
- Throw a die 600 times and count the number of 1s, 2s, etc

Category	Observed
1	115
2	97
3	91
4	101
5	110
6	186

Chi-square Distributions: Its Purpose

What is the expected number of 1s, 2s etc?

Chi-square Distributions: Its Purpose

What is the expected number of 1s, 2s etc?

Category	Observed	Expected
1	115	100
2	97	100
3	91	100
4	101	100
5	110	100
6	186	100

This is an example where we have an expected outcome and the corresponding observed outcome.

We'd like to know if the die is loaded or not?

Chi-square Distributions: Its Purpose

How might we compare the observed and expected? If they are the same then the difference will be zero so we could compute:

$$O_i - E_i$$

where O_i is the observed and E_i is the corresponding expected. We could sum over the square terms to eliminate negative terms to give:

$$\sum_{i=1}^{n}(O_i - E_i)^2$$

Chi-square Distributions: Its Purpose

The final thing to do is normalize by dividing by E_i to yield:

$$\sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

That is we reduce the relative importance of large expected values to prevent them from dominating the sum.

Note that the smaller the number the better the fit.

We state that this measure is distributed according to a chi-square distribution.

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

with degrees of freedom $k - 1$.

Chi-square Distributions: Its Purpose

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Think of it this way, take the square root on both sides:

$$x_i = \frac{O_i - E_i}{\sqrt{E_i}}$$

If x_i is a random variable that is normally distributed then x_i^2 will be χ^2 distributed.

Chi-square Distributions: Its Purpose

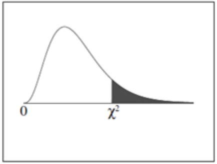
Given H_o that there is no difference between the expected and observed, the larger the value of χ^2 the stronger the evidence against H_o.

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Chi-square Distributions: Tables

As expected there are tables that describe the χ^2 distribution.

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955

Chi-square: Example

Category	Observed	Expected
1	115	100
2	97	100
3	91	100
4	101	100
5	110	100
6	186	100

$$\chi^2 = \frac{(115-100)^2}{100} + \ldots = 6.12$$

We need to determine the p-value at say a critical value of 0.05 (95%). $df = 6 - 1 = 5$. Look up table with df = 5 and column $\chi^2_{0.05}$ yields 11.07. This is bigger than 6.12 which means that 6.12 lies inside the H_o areas. We conclude that the pattern of die throws is not unusual and therefore the die is probably fair.

Chi-square: Limitation

Only use the χ^2 test when ever all the expected values are greater than or equal to 5. In the case of the die they were.

Chi-square: A Classic Example

Mendel's Experiments

A F_1 cross (heterozygous cross) yields 355 yellow and 123 green peas. The expected ratio is 3:1. Total number of peas = 478

H_o: There is no difference between the results of the experiment and the expected ratio.

Expected yellow peas:

$E_1 = 478 \times 3/4 = 385.5$

$E_2 = 478 \times 1/4 = 119.5$

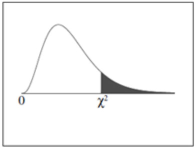
	Observed	Expected
Yellow	355	385.5
Green	123	119.5

Chi-square: A Classic Example

$$\chi^2 = \frac{(355 - 358.5)^2}{385.5} + \frac{(123 - 119.5)^2}{119.5} = 0.137$$

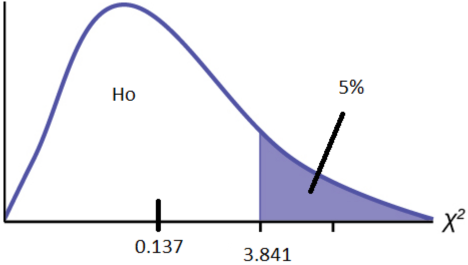
Choose $\alpha = 0.05$ (One tailed test).

$$\text{df} = 2 - 1 = 1$$



$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$
0.004	0.016	2.706	3.841
0.103	0.211	4.605	5.991
0.352	0.584	6.251	7.815
0.711	1.064	7.779	9.488
1.145	1.610	9.236	11.070
1.635	2.204	10.645	12.592
2.167	2.833	12.017	14.067
2.733	3.490	13.362	15.507

Chi-square: A Classic Example



Therefore we do no reject H_o.

Class Exercise

- M&M come in six colors, Red, Orange, Yellow, Green, Blue, and Brown
- 600 M&Ms were obtained by purchasing a bunch of M&M bags and counting the individual colors. The following counts for each of the colors was obtained:

Color	Observed	Expected
Red	115	
Orange	95	
Yellow	120	
Green	105	
Blue	90	
Brown	118	
Sum	600	

- Determine whether bags of M&Ms are filled with equal amounts of each color.

Class Exercise

- H_o : evenly distributed. $p_i = 1/6$.

Color	Observed	Expected
Red	115	100
Orange	95	100
Yellow	120	100
Green	105	100
Blue	90	100
Brown	118	100
Sum	600	

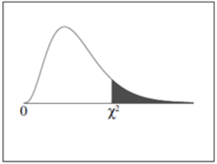
- $\chi^2 = \frac{(115-100)^2}{100} + \frac{(95-100)^2}{100} + \frac{(120-100)^2}{100} + \frac{(105-100)^2}{100} + \frac{(90-100)^2}{100} + \frac{(118-100)^2}{100} = 10.99$
- Degrees of freedom $= 6 - 1 = 5$

Chi-square Distributions: Tables

Look up 10.99 in the body of the χ^2 table. Note that it lies between 10% and 5%, that is the p-value is inside the H_o region. Therefore we do not reject H_o.

The actual p-value $= 0.052$

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

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Chi-square Distributions: Tables

Computing the exact p-value using Python:

```
from scipy import stats
pValue = 1 - stats.chi2.cdf (10.99, 5)
print pValue
0.051578614910744669
```

Restrictions

- All observations must be independent
- Expected counts should not be less than 5, eg Blue $= 2$ although if the number of categories is large some ($< 20\%$) can be less than 5. It the last entries in the table are less than 5 then you should pool them (see next example).
- All counts must be > 0
- Data should be frequency data, variables should be categorical.
- eg Heights are not categorical but you can use ranges:

Range	Height
short (< 5)	10
Middle $5 \leftrightarrow 6$	50
Tall > 6	20

A Harder Problem

- A factory makes prosthetic limbs but it has been found that the manufacturing process produces defects. The factory owners want to know if the defects are purely random or whether there is some systematic non-random process such as a defective machine, or poor workmanship that is causing the defects.

  A random sample of $n = 60$ has been collected for inspection. The following data is the result:

Number of Defects	Observed Frequency
0	32
1	15
2	9
3	4

- Show that the distribution of defects is purely random.

Poisson Distribution

- The number of cars that pass under a road bridge during a given period of time.
- The number of spelling mistakes while typing a single page.
- The number of phone calls at a call center per minute.
- The number of times a web server is accessed per minute.
- The number of animals killed per unit length of road.
- Number of mutations per 100,000 base-pairs on DNA after a certain amount of radiation.
- The number of pine trees per unit area of mixed forest.
- The number of stars in a given volume of space.
- The number of soldiers killed by horse-kicks each year in each corps in the Prussian cavalry.
- The number of light bulbs that burn out in a certain amount of time.
- The number of viruses that can infect a cell in cell culture.
- The number of inventions invented over a span of time in an inventor's career.
- Number of particles that "scatter" off of a target in a nuclear experiment.
- The number of hurricanes in a year that originate in the Atlantic ocean.
- Number of defects per manufactured item.

A Harder Problem

- If the number of defects follows a Poisson distribution then the events should be random.
- Our H_o is therefore that the number of defects follows a Poisson distribution (95%)
- There is a single parameter for a Poisson distribution, the mean rate.

The mean rate of defects per prosthetic limb is:

$$\lambda = (32 \times 0 + 15 \times 1 + 9 \times 2 + 4 \times 3)/60 = 0.75$$

We can use the mean to compute the **expected** frequency of defects.

A Harder Problem

$$f(x) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Num Defects	Probability	Expected defects ($p \times 60$)
0	$(0.75)^0 e^{-0.75}/0! = 0.472$	28.32
1	0.354	21.24
2	0.133	7.98
3	0.041	2.46

A Harder Problem

Expected Defects	Observed	Expected defects
0	32	28.32
1	15	21.24
2	9	7.98
3	4	2.46

Pool the last two rows (< 5)

Expected Defects	Observed	Expected defects
0	32	28.32
1	15	21.24
2	13	10.44

df $= 3 - 1 = 2$ $\quad \alpha = 0.05$

A Harder Problem

Expected Defects	Observed	Expected defects
0	32	28.32
1	15	21.24
2	13	10.44

$$\chi^2 = \frac{(32 - 28.32)^2}{28.32} + \frac{(15 - 21.24)^2}{24.24} + \frac{(13 - 10.44)^2}{10.44} = 2.94$$

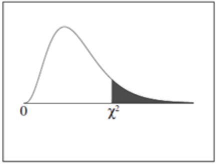
A Harder Problem

Look up 2.94 in the body of the χ^2 table. Note that it lies between 0.9 and 0.1, that is the p-value is well within the H_o region. Therefore we do not reject H_o. There is no evidence to suggest that the defects are non-random.

The actual p-value = 0.229

The 5% cutoff point is 5.99 (compare to 2.94)

Chi-Square Distribution Table



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6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548

Chi-square Distributions: Contingency Tables

What if we had various types of categories and not just one?
The text book cites the example of the manufacture of steel pins made by different machines. We can describe the pins either as too thin, ok or too thick.

	Too Thin	OK	Too Thick	Total
Machine 1	10	102	8	120
Machine 2	34	161	5	200
Machine 3	12	79	9	100
Machine 4	10	60	10	80
Total	66	402	32	500

These are called contingency or two-way tables.

Chi-square Distributions: Contingency Tables

	Not a Smoker	Smoker	Total
Male	96	39	135
Female	87	18	105
Total	183	57	240

For the columns you could have examples such as left or right-handed, or blond, brunette, red-head, Mouse colored hair, or six different genetic markers, or tall, medium or short height etc.

Chi-square Distributions: Contingency Tables

	Cancer	Fatal Heart Disease	Non-fatal Heart Disease	Healthy	Total
Healthy Diet	15	24	25	239	303
Mediterranean	7	14	8	273	302
Asian	4	3	16	260	283
Burger Diet	21	39	45	190	295
Total	47	80	94	962	1183

Check Diez for further examples.

Chi-square Distributions: Contingency Tables

In general:

	Column 1	Column 2	...	Column J	Total
Row 1	O_{11}	O_{12}	...	O_{1J}	O_1
Row 2	O_{21}	O_{22}	...	O_{2J}	O_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Total	T_1	T_2	...	T_J	T_T

Contingency tables can be used to test if there is a relationship between the rows and columns.

For example, does diet affect the likelihood of getting heart disease or cancer, or is diet immaterial?

Contingency Tables: Example

H_o: The probability that the outcome of a trial falls into column j is the same for each row i. i.e it does not matter what row we pick, the probability of the outcomes in the rows are the same (eg diet does not matter).

How do we compute these probabilities assuming there is not effect? These are the expected outcomes.

Calculate Expected (E) Numbers

	Chocolate	Strawberry	Vanilla	Total
Female	70	35	60	165
Male	30	32	47	109
Total	100	67	107	274

$\frac{100}{274}$ or 36.5% liked chocolate

If there is no difference between males and females how many females would you expect to like chocolate?

There are 165 females in all therefore we would expect 36.5% of 165 females to like chocolate $= 60.2$ females.

By subtraction it must be the case that $100 - 60.2$ makes would like chocolate $= 39.8$

Calculate Expected (E) Numbers

We do the same for the other entries.

Expected Numbers:

	Chocolate	Strawberry	Vanilla	Total
Female	60.2	40.4	64.4	165
Male	39.8	26.6	42.6	109
Total	100	67	107	274

If you want the formula then:

$$\text{Expected Count} = \frac{\text{Row Total} \times \text{Column Total}}{\text{Overall Total}}$$

We now have a set of observed and expected numbers and we can carry out a chi-square test. We assume that the underlying distribution is normal.

Calculate Expected (E) Numbers

Degrees of freedom:

$$= (\text{nRows} - 1) \times (\text{nColumns} - 1) = (2 - 1)(3 - 1) = 2$$

$$\chi^2 = \sum_i \sum_j \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

The χ^2 is summed over every entry in the table (except the totals)
For the ice-cream table:

$$\chi^2 = 6.58$$

The area to the left this represents $= 0.037$

This is less than 0.05 therefore we reject H_o. There is evidence to suggests that males and females have different preferences for ice-cream.

Try this Example

Hypertension	Low Fat Diet	Average Fat Diet	High Fat Diet	Total
Yes	24	33	46	103
No	109	101	87	297
Total	133	134	133	400

Is there a relationship between a fat diet and hypertension (high blood pressure)?

F Test for Equality of Variance

Test for variance equality.

F Test for Equality of Variance

What if you want to know if the variance of two independent populations is the same?

This will become more important when we study the analysis of variance.

Let $X_1, X_2, \ldots, X_m$ be a sample from $N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, \ldots, Y_n$ from $N(\mu_2, \sigma_2^2)$. Assume both samples were selected independently. The values for μ_1 and μ_2 are unimportant.

Then if

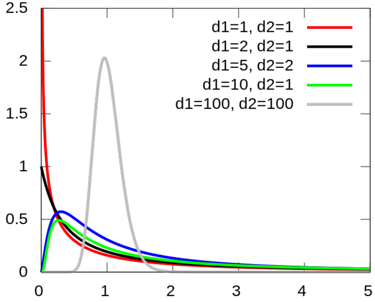
$$F = \frac{\sigma_1^2}{\sigma_2^2}$$

the ratio has a F distribution.

$$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\frac{\nu_1}{\nu_2})^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2}-1}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})(1+\frac{\nu_1 x}{\nu_2})^{\frac{\nu_1+\nu_2}{2}}}$$

F Test for Equality of Variance

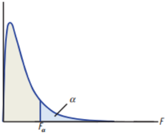
There are two degrees of freedom, one for the numerator $(n_1 - 1)$ and another for the denominator $(n_2 - 1)$:



$F_{3,5}$ means an F distribution with 3 degrees of freedom for the numerator and 5 degrees of freedom for denominator.

F Distribution: Table

Critical Values of the *F*-Distribution ($\alpha = 0.025$)



Numerator Degrees of Freedom

Denominator Degrees of Freedom

	1	2	3	4	5	6	7	8	9
1	647.7890	799.5000	864.1630	899.5833	921.8479	937.1111	948.2169	956.6562	963.2846
2	38.5063	39.0000	39.1655	39.2484	39.2982	39.3315	39.3552	39.3730	39.3869
3	17.4434	16.0441	15.4392	15.1010	14.8848	14.7347	14.6244	14.5399	14.4731
4	12.2179	10.6491	9.9792	9.6045	9.3645	9.1973	9.0741	8.9796	8.9047
5	10.0070	8.4336	7.7636	7.3879	7.1464	6.9777	6.8531	6.7572	6.6811
6	8.8131	7.2599	6.5988	6.2272	5.9876	5.8198	5.6955	5.5996	5.5234
7	8.0727	6.5415	5.8898	5.5226	5.2852	5.1186	4.9949	4.8993	4.8232
8	7.5709	6.0595	5.4160	5.0526	4.8173	4.6517	4.5286	4.4333	4.3572
9	7.2093	5.7147	5.0781	4.7181	4.4844	4.3197	4.1970	4.1020	4.0260
10	6.9367	5.4564	4.8256	4.4683	4.2361	4.0721	3.9498	3.8549	3.7790
11	6.7241	5.2559	4.6300	4.2751	4.0440	3.8807	3.7586	3.6638	3.5879
12	6.5538	5.0959	4.4742	4.1212	3.8911	3.7283	3.6065	3.5118	3.4358
13	6.4143	4.9653	4.3472	3.9959	3.7667	3.6043	3.4827	3.3880	3.3120
14	6.2979	4.8567	4.2417	3.8919	3.6634	3.5014	3.3799	3.2853	3.2093
15	6.1995	4.7650	4.1528	3.8043	3.5764	3.4147	3.2934	3.1987	3.1227
16	6.1151	4.6867	4.0768	3.7294	3.5021	3.3406	3.2194	3.1248	3.0488
17	6.0420	4.6189	4.0112	3.6648	3.4379	3.2767	3.1556	3.0610	2.9849
18	5.9781	4.5597	3.9539	3.6083	3.3820	3.2209	3.0999	3.0053	2.9291
19	5.9216	4.5075	3.9034	3.5587	3.3327	3.1718	3.0509	2.9563	2.8801
20	5.8715	4.4613	3.8587	3.5147	3.2891	3.1283	3.0074	2.9128	2.8365
21	5.8266	4.4199	3.8188	3.4754	3.2501	3.0895	2.9686	2.8740	2.7977
22	5.7863	4.3828	3.7829	3.4401	3.2151	3.0546	2.9338	2.8392	2.7628
23	5.7498	4.3492	3.7505	3.4083	3.1835	3.0232	2.9023	2.8077	2.7313
24	5.7166	4.3187	3.7211	3.3794	3.1548	2.9946	2.8738	2.7791	2.7027
25	5.6864	4.2909	3.6943	3.3530	3.1287	2.9685	2.8478	2.7531	2.6766
26	5.6586	4.2655	3.6697	3.3289	3.1048	2.9447	2.8240	2.7293	2.6528

F Distribution: Table

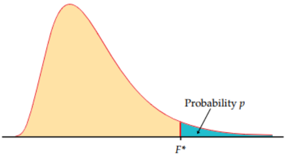


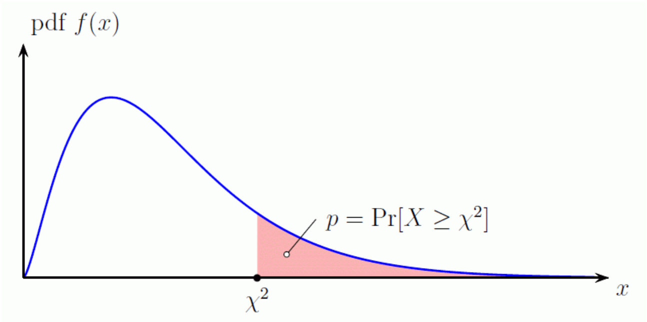
Table entry for p is the critical value F^* with probability p lying to its right.

TABLE E

F critical values

Degrees of freedom in the denominator	p	Degrees of freedom in the numerator 1	2	3	4	5	6	7	8	9
1	.100	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86
	.050	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
	.025	647.79	799.50	864.16	899.58	921.85	937.11	948.22	956.66	963.28
	.010	4052.2	4999.5	5403.4	5624.6	5763.6	5859.0	5928.4	5981.1	6022.5
	.001	405284	500000	540379	562500	576405	585937	592873	598144	602284
2	.100	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
	.050	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
	.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39
	.010	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
	.001	998.50	999.00	999.17	999.25	999.30	999.33	999.36	999.37	999.39
3	.100	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
	.050	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47
	.010	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
	.001	167.03	148.50	141.11	137.10	134.58	132.85	131.58	130.62	129.86
4	.100	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
	.050	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90
	.010	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
	.001	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47
5	.100	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
	.050	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
	.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
	.010	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
	.001	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	27.24

F Distribution: Table



F Test for Equality of Variance

Types of hypotheses that can be tested:

$$H_o : \frac{\sigma_1^2}{\sigma_2^2} \leq 1$$

$$H_o : \frac{\sigma_1^2}{\sigma_2^2} \geq 1$$

$$H_o : \frac{\sigma_1^2}{\sigma_2^2} = 1 \quad \text{means} \quad \sigma_1^2 = \sigma_2^2$$

We'll focus on the equality test.

F Test for Equality of Variance: Example

Consider two samples ($n_1 = 12$ and $n_2 = 14$) drawn from two normal populations. Assume that the variance for the two samples is:

$$s_1^2 = 5 \quad s_2^2 = 11$$

Our H_o is $s_1^2 = s_2^2$ and H_1 is $s_1^2 \neq s_2^2$

Choose a significance level of 0.05.

Because we're dealing with an equality test we must use a two-tailed test.

Compute the ratio of the variances:

$$F = \frac{s_1^2}{s_2^2}$$

Most tables only give the right-hand tail. Therefore arrange the ratio so that the larger variance is in the numerator. This ensures that the ratio will be > 1 and therefore the F statistics will be located on the right-had side of the distribution.

F Test for Equality of Variance: Example

$$F = \frac{s_1^2}{s_2^2} = \frac{11}{5} = 2.2$$

Next look up the table.

$$df_{num} = 14 - 1 \quad df_{den} = 12 - 1$$

Recall the significance level was set to 0.05 but we're doing a two tailed test, but tables only give the right hand tail. Therefore the we need to halve the significance level to 0.025.

We must therefore use a table with a significance level of 0.025.

F Test for Equality of Variance: Example

$$F = 2.2 \quad \mathsf{df}_{\mathsf{num}} = 13 \quad \mathsf{df}_{\mathsf{den}} = 11$$

0.025

see below for more

Degees of Freedom of the numerator — df1

df2	11	12	13	14	15	16	17	18	19	20	df2
1	973.0284	976.7246	979.8387	982.5453	984.8736	986.9109	988.7153	990.3451	991.8003	993.0809	1
2	39.40659	39.41477	39.42114	39.4266	39.43114	39.43569	39.43933	39.44206	39.44569	39.44751	2
3	14.37411	14.33659	14.30453	14.27679	14.25269	14.23155	14.21267	14.19608	14.18084	14.16743	3
4	8.793563	8.751158	8.715006	8.683742	8.656571	8.632583	8.611323	8.592338	8.575284	8.559937	4
5	6.567802	6.524544	6.487596	6.455593	6.42774	6.403184	6.381356	6.361915	6.344408	6.328548	5
6	5.40976	5.366246	5.329014	5.296812	5.268646	5.243862	5.221807	5.202139	5.184404	5.168403	6
7	4.709477	4.665822	4.628475	4.596075	4.567795	4.542812	4.520643	4.500777	4.482899	4.466756	7
8	4.243418	4.199677	4.16216	4.129674	4.101224	4.076099	4.05376	4.033751	4.01576	3.999446	8
9	3.912078	3.868223	3.830593	3.797965	3.769344	3.744105	3.721624	3.701473	3.68334	3.666912	9
10	3.664923	3.620954	3.583182	3.550412	3.521677	3.496268	3.473644	3.45338	3.435105	3.418535	10
11	3.473701	3.429619	3.391733	3.358821	3.329944	3.304393	3.281642	3.261235	3.242832	3.226148	11
12	3.321475	3.277279	3.239265	3.20621	3.177206	3.151527	3.128633	3.108113	3.089582	3.07277	12
13	3.197499	3.153175	3.115034	3.081851	3.052719	3.026912	3.00389	2.983242	2.964583	2.947672	13
14	3.094584	3.050161	3.011891	2.978595	2.949321	2.9234	2.900265	2.879489	2.860716	2.843692	14
15	3.007827	2.963276	2.924907	2.891483	2.862095	2.836046	2.812797	2.791907	2.773035	2.755897	15
16	2.933703	2.889053	2.850555	2.817018	2.787516	2.761354	2.738005	2.717002	2.69803	2.680792	16
17	2.869641	2.824891	2.786294	2.752643	2.723027	2.696765	2.673303	2.6522	2.633129	2.615799	17
18	2.813735	2.768886	2.730189	2.696424	2.666724	2.640348	2.616787	2.595598	2.576428	2.559005	18
19	2.764523	2.719574	2.680778	2.646928	2.617114	2.590653	2.566992	2.545704	2.526448	2.508941	19
20	2.720867	2.675833	2.636938	2.603002	2.573103	2.546543	2.522789	2.501416	2.482075	2.464489	20

F Test for Equality of Variance: Example

The critical value is 3.39 at 2.5%

However the F value of 2.2 is smaller than 3.39, therefore we are within the H_o region.

We conclude that at the 5% level there is sufficient evidence to cast doubt on the hypothesis that the two variances are equal.

Computing F distribution area using Python

Computing the exact p-value using Python:

```
from scipy import stats
pValue = stats.f.sf(2.2, dfn=11, dfd=13)
print pValue
0.089024550297109858
```

0.089 is greater than 0.025 therefore we do not reject H_o.