Counting[1]

October 17, 2017

[1]HMS, 2017, v1.2

Chapter References

- Navidi, Chapter 2.2

Motivation

- Many basic probability problems are counting problems
  - Example: Assume there is 1 man and 2 women in a room. You pick a person randomly. What is the probability P1 that this is a man? If you pick two persons randomly, what is the probability P2 that these are a man and woman
  - Answer: These are the possible outcomes: (M), (W1), (W2) so

$$P_1 = \frac{\#\text{ successful events}}{\#\text{ events}} = \frac{\#\text{ men}}{\#\text{ men} + \#\text{ women}} = \frac{1}{3}$$

  - To compute P_2 we can write out all the possible events: (M,W1), (M,W2), (W1,W2) so

$$P_2 = \frac{\#\text{ successful events}}{\#\text{ events}} = \frac{2}{3}$$

  - Both problems consist of counting the number of different ways that certain events can occur

Another Example

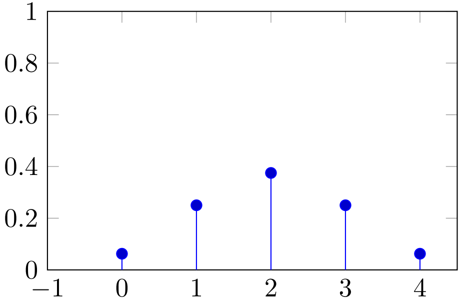
In class we described the probability distribution of the number of heads when flipping 4 coins. This was done by enumerating all possible arrangements of heads and tails in four coins: TTTT, HHHH, THHH, etc. From this data the probability distribution was computed as:

Random Variable: X	0	1	2	3	4
$p(x)$	1/16	4/16	6/16	4/16	1/16

Another Example

$X = 0$	$X = 1$	$X = 2$	$X = 3$	$X = 4$
		HTTH		
		HTHT		
	HTTT	THTH	HHHT	
	THTT	HHTT	HHTH	
	TTHT	THHT	HTHH	
TTTT	TTTH	TTHH	THHH	HHHH
1/16	4/16	6/16	4/16	1/16

Probability Mass Function: PMF



Heights represent probabilities.

Counting Rule 1

- Definitions: $n! = \prod_{k=1}^{n} k \quad 0! = 1$
- If an operation can be done in n ways, and we carry out N trials, the number of arrangements will be:

$$n^N$$

- **example**: For a coin: n $= 2$, we throw $N = 4$, therefore number of arrangements $= 2^4 = 16$
- **example**: For a 6-sided die thrown two times, $n = 6, N = 2$, therefore the number of arrangements $= 6^2 = 36$

Counting Rule 2

- Generalization of Rule 1 where each trial can have different ways.
- If trial 1 has K_1 ways, trial 2 has K_2 ways, etc then the number of arrangements is:

$$K_1 K_2 K_3 \dots$$

- **example**: Throw a coin followed by a six-sided die:
$K_1 = 2, K_2 = 6$, therefore number of arrangements $= K_1 K_2 = 12$
- **example**: Licence plates in Washington State, eg 954 ZDZ. That is $K_1 = 10, K_2, = 10, K_3 = 10, K_4 = 26, K_5 = 26, K_6 = 26$. The number of arrangements will therefore be:

$$10^3 \ 26^3 = 17,576,000$$

Counting Rule 3: Permutations

- The number of ways n distinct things can be arranged in order is $n!$.
- **example**: ABC, $n = 3$, therefore the number of arrangements $= 3! = 6$
- **example**: ABC146, $n = 6$, therefore the number of arrangements $= 6! = 720$
- ORDER IS IMPORTANT

Counting Rule 4: Generalization of Permutations

- The number of ways of arranging r objects from n distinct objects is given by:

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

- **example**: How many ways is it possible to arrange 2 letters out of ABCDE?

$$^{n}P_{r} = \frac{5!}{(5-2)!} = \frac{120}{6} = 20$$

- ORDER IS IMPORTANT, ie AB is different from BA

Counting Rule 5: Combinations

- The number of ways of arranging r objects from n distinct objects irrespective of order is given by:

$$nC^r = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

- The term $\binom{n}{r}$ is called the **binomial coefficient.**

- **example**: How many ways is it possible to arrange 2 letters out of ABCDE where the order is unimportant?

$$nC^r = \frac{5!}{(5-2)!2!} = \frac{120}{12} = 10$$

AB, AC, AD, AE, BC, BD, BE, CD, CE, DE

- ORDER IS NOT IMPORTANT, ie AB is NO different from BA

Counting Rule 5: Combinations

- Some algebraic relations of binomial coefficients:

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{r} = \binom{n}{n-r}$$

- **example**: $\binom{10}{3} = \binom{10}{7}$ or $\binom{50}{49} = \binom{50}{1}$

Combinations: Lottery Problem

- A lottery consists of picking 6 numbers from a lottery card that lists numbers 1, 2, 3, 4, 5, etc all the way to 49.
- At the lottery, 6 winning numbers are picked out.
- What is the chance of winning the lottery?
- First, how many ways are there to pick 6 distinct numbers, where the order doesn't matter? In other words how many combinations of 6 distinct numbers are there?

$$nC^r = \frac{49!}{(49-6)!6!} = 13,983,816$$

- Probability of winning the lottery is $1/13,983,816 = 7.15112 \times 10^{-8}$. Or 0.999999928 of not winning.

Examples

Vasopressin (Retains water in the body and constricts blood vessels) is a small polypeptide composed of 9 distinct amino acids in a particular order. The order of these amino acids is of critical importance to the proper functioning of vasopressin. If these 9 amino acids were placed in a hat and drawn out randomly one by one, how many different arrangements of these 9 amino acids would be possible?

Order is important, objects are distinct, therefore compute permutations, no subset, therefore use simple version of permutations, rule 3:

$$9! = 362,880$$

Examples

Solution: Let A, B, C, D, E, F, G, H, I symbolize the 9 amino acids. They must fill 9 slots.

There are 9 choices for the first position, leaving 8 choices for the second slot, 7 choices for the third slot and so on. The number of different orderings is therefore

$$9(8)(7)(6)(5)(4)(3)(2)(1) = 9! = 362,880$$

Examples

Given 20 amino acids, how many different dipeptides can be formed? Assume for now that each dipeptide has to have two **distinct** amino acids.

Order is important, objects are distinct, we're considering a subset (2 amino acids), this is therefore a general permutation problem:

$$^{n}P_{r} = \frac{20!}{(20-2)!} = 380$$

If we allow dipeptides to contain identical amino acids then there is an additional 20 combinations to consider, bringing the total to 400.

Examples

Another way to think of the last example is where we allow a dipeptide to have the same amino acid is to invoke Rule 1.

If an operation can be done in n ways, and we carry out N trials, the number of arrangements will be:

$$n^N$$

$n = 20, N = 2$, therefore number of arrangements $= 20^2 = 400$

Examples

Using rule 1 we can also solve the following problem. Given 20 amino acids how many different proteins can be made that are 300 amino acids long?

$n = 20, N = 300$, therefore number of arrangements

$$= 20^{300} = 4.63221192402 \times 10^{395}$$

Number of atoms in the Universe is roughly 10^{60}.

Examples

- On a book shelf there are 10 books. 3 of the books are biology (B), 2 of the books are chemistry (C), 4 of the books are physics (P) and 1 book is on mathematics (M).
- You don't care how the books are arranged except that similar topics must be kept together, for example grouping the books as PCBM, or CBMP is ok.
- Within a group you don't care how they are arranged. How many different arrangements are there?
- For the groups there are 4! ways to arrange them $= 24$.
- For a given group of n books there are $n!$ arrangements, eg $3! = 6$ ways to arrange the biology books.
- Therefore total number of arrangements is $(3!\ 2!\ 4!\ 1!)24 = 6912$

Counting Rule 6: Permutation rule with repeated objects

- Consider all the permutations of the letters in the word BOB.
- Since there are three letters, there should be $3! = 6$ different permutations. Those permutations are BOB, BBO, OBB, OBB, BBO, and BOB.
- However permutations such as OBB and OBB look the same, ie indistinguishable. There are in fact only three distinguishable permutations, BOB, OBB, and BBO.
- To find the number of distinguishable permutations, we take the total number of indistinguishable permutations and divide by the frequency of each letter factorial. In this case B appears twice, therefore:

$$\frac{6!}{2!} = 3$$

Counting Rule 6: Permutation rule with repeated objects

- A classical example is finding the number of distinguishable permutations of MISSISSIPPI. Put more formally, if a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind and so on, with:

$$n = n_1 + n_2 + n_3 + \dots$$

  then the number of distinguishable permutations of n objects is given by:

$$\frac{n!}{n_1! n_2! n_3! \dots}$$

- **example**: How many ways can the word MISSISSIPPI be arranged? If we don't care about equivalent variants then the number of permutations is $11! = 39,916,800$ However the number of distinguishable permutations is actually

$$\frac{11!}{4!\ 4!\ 2!\ 1!} = 34,650$$

Counting Rule 6: Permutation rule with repeated objects

- 15 cells are trapped in a microfluidic device, 4 cells are found to be dead, how many ways is it possible to arrange the cells?

Counting Rule 6: Permutation rule with repeated objects

- 15 cells are trapped in a microfluidic device, 4 cells are found to be dead, how many ways is it possible to arrange the cells? We have two types of cell, alive and dead, there are therefore two distinct types that are repeated multiple times. We can use rule 6 to solve this problem:

$$\frac{n!}{n_1!\ n_2!\ n_3!\ \ldots} = \frac{15!}{11!\ 4!} = 1365$$

Class Problem

- A cell culture contains two types of cell in roughly equal quantities.
- A researcher picks out 20 cells from the culture randomly, or so they claim.
- It turns out that in all their studies, the researcher picked out 10 cells of the first type and 10 cells of the other type.
- What is the probability that the researcher could have picked 10 cells of each. Is the researcher biased in the way the cells are picked?

Class Problem

- Hints to solving previous problem
- How many possible arrangements are there of 2 types of cell in a collection of 20?
- How many ways are there to arrange 20 cells where there are 10 of each type?
- How likely is it that one of the 10/10 arrangements was picked from the totality of arrangements?

Class Problem

- Answer:

  There are 2^{20} ways to arrange 20 cells.

$$\frac{20!/(10!\ 10!)}{2^{20}} = \frac{184756}{2^{20}} = 0.176$$

Very Important Variation

- Consider the case when we have only two types of symbol. Assume we have r of the first symbol and n symbols in total.
- From the above we can state that there will be $n - r$ of the second symbol.
- As before the number of ways that these symbols can be arranged is:

$$\frac{n!}{n_1!\ n_2!}$$

- That is:

$$\frac{n!}{n1!\ n2!} = \frac{n!}{r!(n-r)!}$$

- In other words, the number of ways to arrange two repeating symbols is the same as the number of combinations.

Important Variation: Example

- Assume we have 7 coins, of which 3 are heads and 4 are tails.
- How many ways can we arrange the heads within the group of 7 coins?

$$\frac{7!}{3!\ 4!} = \frac{7!}{3!(7-3)!} = \frac{5040}{6 \times 24} = 35$$

- HTTTHHT HTTHTHT TTHTHTH HHTTTTH THTTTHH
THHTHTT TTTHTHH HHHTTTT THTTHTH HHTHTTT
HHTTTHT TTHTHHT HTTHHTT HTTTHTH HTTTTHH
TTHHTHT HTHHTTT THTHTTH HTTHTTH THHTTTH
THTHTHT TTHHTTH THHHTTT HTHTHTT HHTTHTT
TTHTTHH THHTTHT TTHHHTT HTHTTHT THTHHTT
TTTHHHT HTHTTTH TTTTHHH THTTHHT TTTHHTH

Code to Generate 35 variations

```
from __future__ import print_function

count = 0
import itertools as it
permset = set([i for i in
      it.permutations(["H","H","H","T","T","T","T"])])
for x in permset:
    str = ''
    for letter in x:
        str = str + letter
    if count % 4 == 0:
        print()
    print (str + ' ', end='')
    count = count + 1
```

Software Support

```
Excel:

Permutations (in EXCEL =PERMUT(6,2))
Combinations (in EXCEL =COMBIN(6,2))

Python:

scipy.special.comb (N, k)
scipy.special.perm (N, k)
```