Linear Combinations of Random Variables[1]

UW

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[1] HMS, 2017, v1.0

Chapter References

- Navidi, Chapter 2.5

Linear Combinations: Adding a Constant

- Adding a Constant value, for example 2.0:

  Consider: 2, 6, 4, 1 $\Rightarrow \mu = 3.25$
  Consider: 4, 8, 6, 2 $\Rightarrow \mu = 5.25$

- Mean has also increased by 2.0. In general:

$$\mu_{X+a} = \mu_X + a \qquad or \qquad E(X+a) = E(X) + a$$

Linear Combinations: Adding a Constant

- Adding a Constant value, for example 2.0:

  Consider: 2, 6, 4, 1 $\Rightarrow \sigma^2 = 3.69$
  Consider: 4, 8, 6, 2 $\Rightarrow \sigma^2 = 3.69$

- Variance is unchanged. In general:

$$\sigma^2_{X+a} = \sigma^2_X \qquad or \qquad Var(X+a) = Var(X)$$

Linear Combinations: Multiply by a Constant

- Multiply by a Constant value, for example 2.0:

  Consider: 2, 6, 4, 1 $\Rightarrow \mu = 3.25$
  Consider: 4, 8, 6, 2 $\Rightarrow \mu = 6.5$

- Mean has change by the same factor. In general:

$$\mu_{aX} = a\mu_X \qquad or \qquad E(aX) = aE(X)$$

Linear Combinations: Multiply by a Constant

- Multiply by a Constant value, for example 2.0:

  Consider: 2, 6, 4, 1 $\Rightarrow \sigma^2 = 3.69$
  Consider: 4, 8, 6, 2 $\Rightarrow \sigma^2 = 14.76$

- Variance has changed by a factor of the square of the constant. In general:

$$\sigma^2_{aX} = a^2\sigma^2_X \qquad or \qquad Var(aX) = a^2Var(X)$$

Linear Combinations: Add and Multiply by a Constant

$$\mu_{aX+b} = a\mu_X + b \qquad or \qquad E(aX+b) = aE(X) + b$$
$$\sigma^2_{aX+b} = a^2\sigma^2_X \qquad or \qquad Var(aX+b) = a^2Var(X)$$

Linear Combinations of Means

- The mean of $X_1 + X_2 + \ldots$ is

$$\mu_{X_1} + \mu_{X_1} + \ldots = \mu_{X_1+X_2+\ldots}$$

- In general:

$$c_1X_1 + c_2X_2 + \ldots = c_1\mu_{X_1} + c_1\mu_{X_1} + \ldots$$

- In expectation form:

$$E(c_1X_1 + c_2X_2 + \ldots) = c_1E(X_1) + c_2E(X_2) + \ldots$$

Linear Combinations of Variances

- The variance of $X_1 + X_2 + \ldots$ is

$$\sigma^2_{X_1} + \sigma^2_{X_1} + \ldots = \sigma^2_{X_1 + X_2 + \ldots}$$

  Note you cannot add standard deviations

- In general:

$$\sigma^2_{c_1 X_1 + c_2 X_2 + \ldots} = c_1^2 \sigma^2_{X_1} + c_2^2 \sigma^2_{X_1} + \ldots$$

- In Var form:

$$Var(c_1 X_1 + c_2 X_2 + \ldots) = c_1^2 Var(X_1) + c_2^2 Var(X_2) + \ldots$$

Application

- Find the variance of a sample of means

$$Var(\overline{X}) = Var\left(\frac{1}{n}\sum X_i\right)$$

- Use $Var(aX) = a^2 Var(X)$:

$$Var(\overline{X}) = \frac{1}{n^2} Var\left(\sum X_i\right)$$

- Assume X_i are identically distributed with a common variance σ^2, then by $Var(X_1 + X_2 + \ldots) = Var(X_1) + Var(X_2) + \ldots$

$$Var(\overline{X}) = \frac{1}{n^2}\sum (Var(X_i)) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

Application

- The variance of the sample of mean is

$$Var(\overline{X}) = \frac{\sigma^2}{n}$$

- The standard deviation of this is called the standard error:

$$\mathsf{SE} = \frac{\sigma}{\sqrt{n}}$$

- Standard deviation measures: Precision (how bunched up your measurements are)
- Standard error measures: Accuracy (assuming there is no fixed bias)

Example

- A researcher dispenses a medium into 96 wells using a manual single tip pipette.
- Ten well plates are selected at random and the mean volume and standard deviation measured:

$$\overline{x} = 1.05 \text{ ml} \qquad \sigma = 0.01 \text{ ml}$$

- The standard error is given by:

$$\text{SE} = \frac{\sigma}{\sqrt{n}} = \frac{0.01}{\sqrt{10}} = 0.0032\text{ml}$$

- Mean $= 1.05 \pm 0.0032$