Discrete Distributions[1]

October 18, 2017

[1] HMS, 2017, v1.0

Chapter References

- Diez: Chapter 3.3, 3.4 (not 3.4.2), 3.5.2
- Navidi, Chapter 4.1, 4.2, 4.3

Discrete Distributions

- Bernoulli – Yes/No Responses
- Binomial Distribution – Sums of Bernoulli Responses
- Geometric – Number of trials until first success
- Poisson – Points in time or space

Bernoulli Distribution

- A Bernoulli event/trial is one where the probability the event occurs is p and the probability the event does not occur is $1 - p$; i.e., the event is has two possible outcomes usually viewed as **success** or **failure**.
- A Bernoulli distribution describes how a Bernoulli event is distributed between success and failure.
- The Bernoulli distribution for a random variable X is often denoted by:

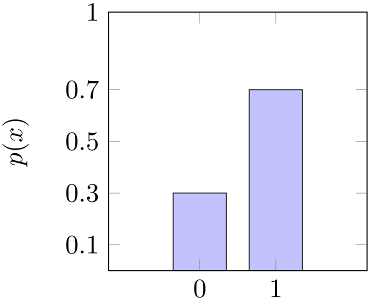
$$X \sim \text{Bernoulli}(p)$$

Bernoulli Distribution

- Examples of Bernoulli Processes:
  - Flipping a coin (again....)
  - Political opinion poll (yes or no to support a candidate)
  - Birth of boy or girl
  - Cell culture grows or does not
  - Ace or King in a pack of cards (with replacement!)
  - Left or right-handedness
  - Whether a drug works or does not

Bernoulli Distribution

- **example**: $X \sim \text{Bernoulli}(0.7)$



Failure and success will often have the values 0 and 1 respectively.

Mean and Variance of a Bernoulli Random Variable

- Recall that the mean is given by $\sum x_i \; p(x_i)$. Therefore the mean of a Bernoulli random variables is:

$$\mu_X = (0)(1-p) + (1)p = p$$

- The variance can be computed similarly using the variance equation

$$\sigma_X^2 = \sum (x_i - \mu_X)^2 p(x_i)$$

Class exercise: Derive the variance

Class exercise: Derive variance

$$\sigma_X^2 = \sum (x_i - \mu_X)^2 p(x_i)$$

- The variance can be computed similarly using the variance equation:

$$\sigma_X^2 = (0 - p)^2(1 - p) + (1 - p)^2(p) = p(1 - p)$$

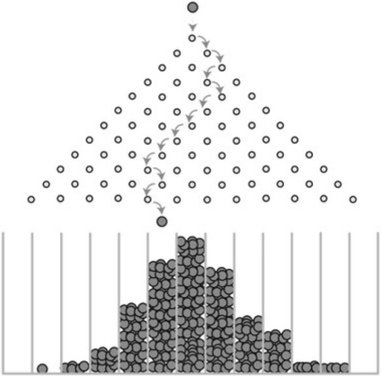
Mean and Variance of a Bernoulli Random Variable

If $X \sim \text{Bernoulli}(p)$, then:

- Mean:
$$\mu_X = p$$
- Variance:
$$\sigma_X^2 = p(1 - p)$$

Galton Board

- Example of something where each trial (row on the board) is a Bernoulli trial.



The Binomial Distribution

- Carry out a Bernoulli trial, and repeat it n times.
- Let the number of successes be a random variable, X
- Then X will have a **Binomial Distribution**.

A random variable X that has a binomial distribution with n trials, and probability p for each trial, is denoted by:

$$X \sim \mathsf{Bin}(n, p)$$

The Binomial Distribution

- A fair coin is tossed 10 times. Let X be the number of heads that appear. What is the distribution of X?

- Since each trial has two outcomes with probability $p = 0.5$, and each trial is independent of the other, then:

$$X \sim \text{Bin}(10, 0.5)$$

Assumptions in a Binomial Distribution

Formal Definition:

- There are only two possible outcomes for each trial, arbitrarily labeled "success" or "failure".
- The trials are **independent** of each other.
- The probability of success is the **same** for each trial.
- There are a **fixed number** n of repeated trials of a simple experiment.

Binomial Distribution

- Toss a coin three times and record the result.
- Question:

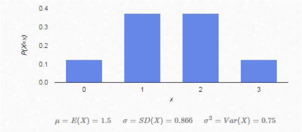
  How many possible arranges of the coin could there be?

Binomial Distribution

- There will be $2^3 = 8$ possible arrangements:
- `HHH, TTT, HHT, HTH, THH, THT, TTH, HTT`
- Plot the distribution of the number of heads:

Number of heads, X:	0	1	2	3
Number of heads observed:	1	3	3	1

- The binomial distribution will predict the above table



Deriving the PMF for the Binomial Distribution

- The distribution of the number heads (or successes), $p(k)$ can be shown to be:

$$p(k) = \binom{n}{k} q^k (1-q)^{n-k}$$

- Where

$$q = \text{probability of success}$$
$$n = \text{number of trials}$$
$$k = \text{The probability we seek for } k \text{ number of heads}$$

Binomial Distribution: Example

Assumptions:

1. n identical trails are performed.
2. There are two outcomes, success and failure in each trial.
3. The trials are independent.
4. The success probability, q, remains constant from trial to trial.

Steps:

1. Identify the success.
2. Determine the probability of success, q.
3. Determine, n, the number of trials.
4. Apply the binomial formula for the number of success we seek.

Binomial Distribution: Example

- **example:** Given five coins what is the probability of throwing 2 heads?

$$p(k) = \binom{n}{k} q^k (1-q)^{n-k}$$

- Set

$$q = 0.5$$
$$n = 5$$
$$k = 2$$

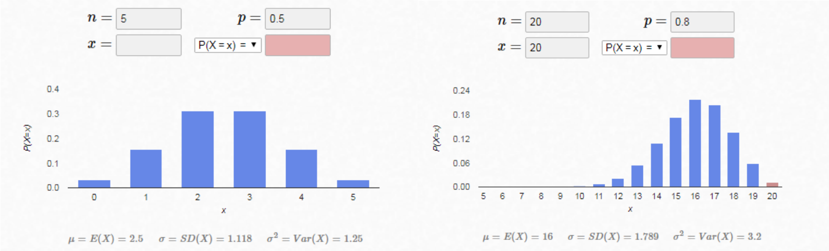
$$p(2) = \binom{5}{2} 0.5^2 (1-0.5)^{5-2} = 10 \times 0.25 \times 0.125 = 0.3125$$

Binomial Distribution: Example

Given the answer of 0.3125 from the last slide, what does it mean?

Deriving the PMF for the Binomial Distribution

- **examples:**



http://homepage.divms.uiowa.edu/~mbognar/applets/bin.html

Deriving the PMF for the Binomial Distribution

- Throw three coins
- Consider $p(0)$, i.e the probability of **not seeing** a single head.
- Let q equal the probability of throwing a head in a single trial.
- Probability of not throwing a head $= (1 - q)$
- Therefore probability of throwing three tails (not three heads):

$$p(0) = (1 - q)^3$$

Deriving the PMF for the Binomial Distribution

- Consider $p(1)$, ie the probability of seeing a single head?
- For example what is the probability of throwing HTT?
- Let q equal the probability of throwing a head in a single trial.
- Probability of **not** throwing a head $= (1 - q)$
- Therefore probability of throwing one head and two tails:

$$p(1) = q(1 - q)^2$$

Deriving the PMF for the Binomial Distribution

- Pattern:

$$p(0) = q^0(1-q)^3$$
$$p(1) = q^1(1-q)^2$$
$$p(2) = q^2(1-q)^1$$
$$p(3) = q^3(1-q)^0$$

- In general, if we have n trials and k is the number of heads then:

$$p(k) = q^k(1-q)^{n-k}$$

- Or is it that simple.....

Deriving the PMF for the Binomial Distribution

- Consider $p(1)$ more closely:
- $p(1)$ is the probability of seeing one head in a toss of three coins and it was $q(1-q)^2$, e.g HTT
- However we know that there are three possible ways to get one head and two tails: `HTT, THT, TTH`.
- Therefore the probability of seeing any of the three patterns (HTT or THT or TTH) is the sum of the three corresponding probabilities:

$$p(1) = 3q(1-q)^2$$

Deriving the PMF for the Binomial Distribution

- Likewise for the other three $p(0), p(2)$, and $p(3)$.
- In each of these cases there are 1, 3 and 1 way respectively.

Deriving the PMF for the Binomial Distribution

- How can we count the number of configurations?
- We've seen previously that that for a word such as MISSISSIPPI, the number of variations is given by:

$$\frac{n!}{n_1!\, n_2!\, \ldots}$$

- The same applies to a sequence such as HHTT, here the number of arrangements is:

$$\frac{4!}{2!\, 2!} = 6$$

Deriving the PMF for the Binomial Distribution

- Assume we have two types of symbol, eg H and T, of which there are n in total.
- We can define r to be the number of heads (H), therefore the number of tails must be $(n - r)$.
- The arrangement formula can now be written as:

$$\frac{n!}{r!(n-r)!}$$

- The result is of course the number of combinations:

$$\binom{n}{r}$$

Deriving the PMF for the Binomial Distribution

- **example:** How many ways can we arrange two heads using four coins, HHTT, HTHT, etc
- $n = 4$, and $r = 2$ (number of heads)
- Compute the combinations:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{4!}{2!(4-2)!} = \frac{24}{4} = 6$$

- Or two heads and six coins, HHTTTT, THHTTT, etc

$$\frac{n!}{r!(n-r)!} = \frac{6!}{2!(6-2)!} = \frac{720}{3(24)} = \frac{720}{72} = 100$$

Why Combinations?

H_1	H_2		
	H_2	H_3	
H_1		H_3	
	H_2		H_4
		H_3	H_4
H_1			H_4

Deriving the PMF for the Binomial Distribution

- To compute the probability of the number of heads $p(k)$ we write:

$$p(k) = \binom{n}{k} q^k (1-q)^{n-k}$$

Binomial Distribution: Example

- Consider a roulette wheel with 38 slots. You win if the ball lands on a number you picked.
- What is the probability of winning twice in 50 spins?
- Each spin is independent, we have a failure or success, therefore we can describe this using a binomial distribution.
- $n = 50, k = 2, q = 1/38$, therefore:

$$p(2) = \binom{50}{2}(1/38)^2(1 - 1/38)^{50-2} = 0.278$$

Binomial Example

- Each year one in three adults aged 65 years and older have severe falls.
- Find the probability that 3 among 11 adults aged 65 years and older will fall in the following year.
- Fall or not fall, independent but finite number of trials.

$$p(k) = \binom{n}{k} q^k (1-q)^{n-k}$$

$$p(3) = \binom{11}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{11-3} = 0.238$$

Binomial Distribution: Properties

- Mean and Variance

$$\mu = nq$$
$$\sigma^2 = nq(1-q)$$

Binomial Theorem

$$(x+y)^2 = x^2 + 2xy + y^2,$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$$

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Binomial Theorem: Pascal's Triangle

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\binom{1}{1} \\
\binom{2}{0}\binom{2}{1}\binom{2}{2} \\
\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3} \\
\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4} \\
\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5} \\
\cdots
\end{array}
\quad = \quad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \cdots & & & & &
\end{array}
$$

$$(p+q)^n = \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} p^x q^{n-x}$$