

336: Systems and Control
Visualization of Simulations
(v1.1)

Contents

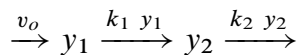
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1 Time Course Simulation

The simplest way to visualize a simulation is to plot the time courses that describe the evolution of the state variables. Although useful, such plots are not the only way to display a behavior of a model. In this chapter we will briefly look at three different ways to visualize a model.

2 Phase Space and Phase Plots

The word **phase space** refers to a space where all possible states are presented. For example, in a biochemical pathway with two species, S_1 and S_2 , the phase space consists of all possible trajectories of S_1 and S_2 in time. For two dimensional systems the phase space can be very conveniently display on an x/y graph where each axis represents one of the state variables. A visual representation of the phase space is often called a **phase portrait** or **phase plane**. To illustrate phase portrait consider the following simple reaction network:



with ODEs:

$$\frac{dy_1}{dt} = v_o - k_1 y_1$$

$$\frac{dy_2}{dt} = k_1 y_1 - k_2 y_2$$

We can assign particular values to the parameters, set up some initial conditions and plot in phase space the evolution of x_1 and x_2 . If we replot the solution using many different initial conditions we get something that looks like the plots shown in Figure 1 to 6.

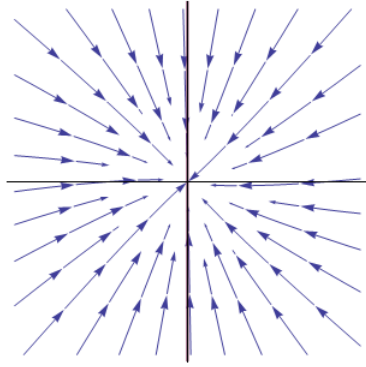
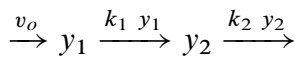


Figure 1: Phase portrait for the two species reaction network. Stable node. **Negative Eigenvalues.** Matrix A : $a_{11} = -2, a_{12} = 0, a_{21} = -0.15, a_{22} = -2$. Corresponding eigenvalues: $\lambda_1 = -2, \lambda_2 = -2$

3 Nullclines

Nullclines are very useful for looking at systems of two variables. If we have a variable x_i with differential equation df_i/dt , then the nullcline is the line that satisfies, $df_i/dt = 0$. If two nullclines are plotted, then their intersection makes the steady state points.

For example, the system:



with ODEs:

$$\frac{dy_1}{dt} = v_o - k_1 y_1$$

$$\frac{dy_2}{dt} = k_1 y_1 - k_2 y_2$$

has two nullclines. Assuming y_1 is plotted on the x axis and y_2 on

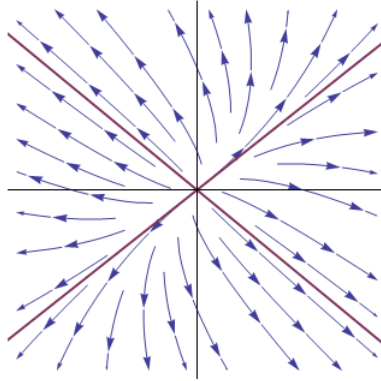


Figure 2: Phase portrait for the two species reaction network. Unstable node, also called an improper node. **Positive Eigenvalues.** Matrix A : $a_{11} = -2, a_{12} = 0, a_{21} = -0.15, a_{22} = -2$. Corresponding eigenvalues: $\lambda_1 = -2, \lambda_2 = -2$

the y axis, then the first nullcline is simply $y_1 = v_o/k_1$, that is a vertical straight line at y_1 . The second equation is given by:

$$y_2 = k_1 y_1 / k_2$$

and represents a straight line with slope k_1/k_2 running through the zero point.

4 Bifurcation Plots

Phase portraits and nullcline visualize time course trajectories. A bifurcation plot is quite different. In a bifurcation plot we visualize how the **steady state solution** changes as a function of a model parameter. So simple models, a bifurcation plot can be quite simple. For example, consider the simple model:

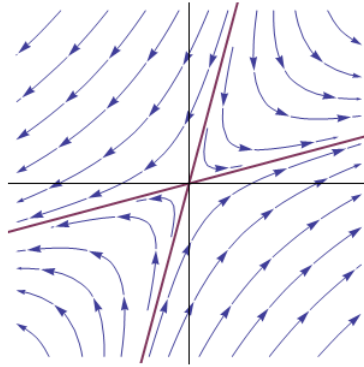
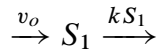


Figure 3: Phase portrait for the two species reaction network. Saddle node. **One Positive and One Negative Eigenvalue.** Matrix A : $a_{11} = 2, a_{12} = -1, a_{21} = 1, a_{22} = -2$. Corresponding eigenvalues: $\lambda_1 = -1.73, \lambda_2 = 1.73$



The differential equation for this is:

$$\frac{dS_1}{dt} = v_o - kS_1$$

The steady state solution can be found by setting the differential equation to zero and solving for S_1 . If we do this we obtain:

$$S_1 = \frac{v_o}{k}$$

To plot a bifurcation plot we must choose a parameter, for example k and plot the steady state solution with respect to k .

Not all bifurcation plots are so simple and later on we will see what bifurcation plots can be so useful when we consider more complex behavior as well as the numerical issues that present themselves in more complex cases.

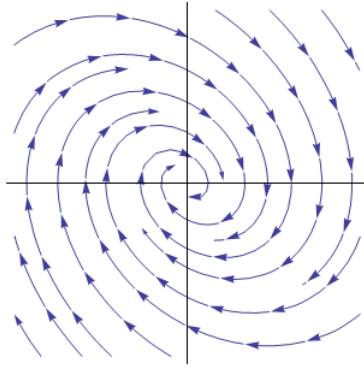


Figure 4: Phase portrait for the two species reaction network. Stable spiral node. **Negative Complex Eigenvalues.** Matrix A : $a_{11} = -0.5, a_{12} = -1, a_{21} = 1, a_{22} = -1$. Corresponding eigenvalues: $\lambda_1 = -0.75 + 0.97i, \lambda_2 = -0.75 - 0.97i$

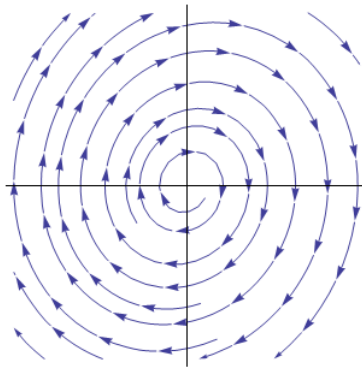


Figure 5: Phase portrait for the two species reaction network. Unstable spiral node. **Positive Complex Eigenvalues.** Matrix A : $a_{11} = 0, a_{12} = 1.0, a_{21} = -1.2, a_{22} = 0.2$. Corresponding eigenvalues: $\lambda_1 = 0.1 + 1.09i, \lambda_2 = 0.1 - 1.09i$

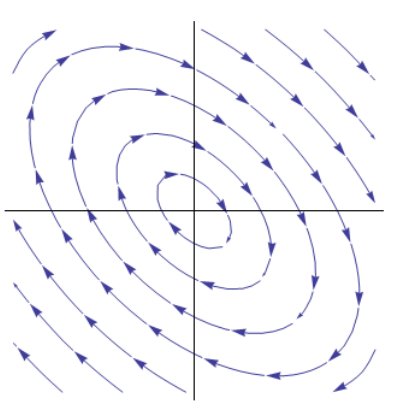


Figure 6: Phase portrait for the two species reaction network. Center node. **Complex Eigenvalues, Zero Real Part.** Matrix A : $a_{11} = 1, a_{12} = 2.0, a_{21} = -2, a_{22} = -1$. Corresponding eigenvalues: $\lambda_1 = 0 + 1.76i, \lambda_2 = 0 - 1.76i$

Description	Eigenvalues	Behavior
Both Positive	$r_1 > r_2 > 0$	Unstable
Both Negative	$r_1 < r_2 < 0$	Stable
Positive and Negative	$r_1 < 0 < r_1$	Saddle point
Complex Conjugate	$r_1 > r_1 > 0$	Unstable spiral
Complex Conjugate	$r_1 < r_1 < 0$	Stable spiral
Pure Imaginary	$r_1 = r_1 = 0$	Center

Table 1: Summary of Node Behaviors

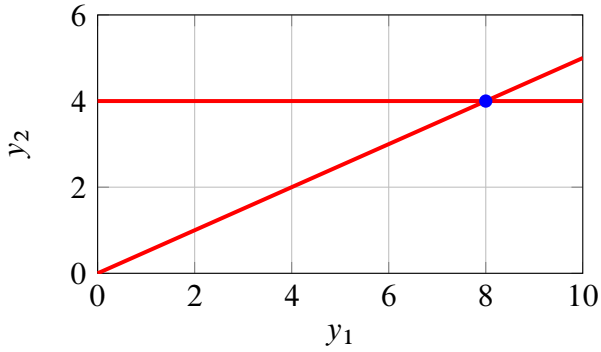


Figure 7: Nullclines for the system: $dy_1/dt = v_o - k_1y_1$; $dy_2/dt = k_1y_1 - k_2y_2$ where $v_o = 6, k_1 = 0.15, k_2 = 3$. The intersection point marks the steady state when both equations equal zero.

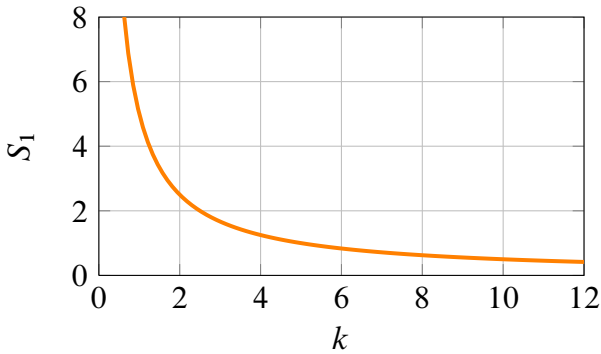


Figure 8: Very simple bifurcation plot system $v_o \xrightarrow{S_1} \xrightarrow{kS_1}$ with respect to k . $v_o = 5$. The graph shows how the steady state value of S_1 changes as a function of k .