336: Systems and Control
Introduction to Laplace Transforms
(v1.31)

Contents

1 Transfer Function

Consider the Laplace transform for the following LTI system:

$$\mathcal{L}\left(\frac{d\boldsymbol{x}}{dt}\right) = \mathcal{L}\left(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\mathbf{u}\right)$$

If we assume assuming zero initial conditions then we can write:

$$\boldsymbol{X}(s) = \left[(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\right]\boldsymbol{U}(s)$$

The term $(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$ is called the **transfer function matrix**, sometimes indicated by $\boldsymbol{H}(s)$:

$$\boldsymbol{X}(s) = \boldsymbol{H}(s)\boldsymbol{U}(s)$$

In deriving this equation we assumed that the initial conditions for the state variables were **zero**. We can therefore define the transfer function as:

The **transfer function** for a given state variable is defined as the ratio of the output, $Y(s)$ to the input, $X(s)$ of the system in the Laplace domain. All initial conditions are assumed to be **zero**.

$$H(s) = \frac{Y(s)}{X(s)}$$

For a given state variable, the transfer function is generally a ratio of two polynomials in s. The powers of s will all be integral powers. Another condition satisfied by physical systems is that the order of the numerator polynomial is less than or equal to the order of the denominator polynomial. It is this property that also allows us to use partial fractions to separate the terms when we wish to apply the inverse Laplace transform.

Since the Laplace transform of the impulse response $\delta(t)$ is 1, we can also state that the output function, $Y(s)$ is equal to the transfer function when we apply an impulse function to the system:

$$Y(s) = H(s) \quad \text{when the input is an impulse function}$$

The denominator in the transfer function is often called the **characteristic polynomial** and the equation formed by setting the denominator to zero is called the **characteristic equation**

In engineering, the roots of the characteristics equation are called the poles, while the roots of the numerator are called the zeros.

2 Convolution

If we have a system where all initial conditions are zero and we apply an impulse function, $\delta(t)$, the output is called the **impulse response** and is represented by the symbol $h(t)$. If the strength of the impulse is k then the response will be $kh(t)$. It is also possible to apply the impulse at a later time, τ, such that an impulse $\delta(t-\tau)$ will give a response $h(t-\tau)$. If we apply two impulses at different times t and τ and with different strengths, k_1 and k_2, the response will be the sum of the individual responses (because the system is linear), that is:

$$k_1 h(t) + k_2 h(t-\tau)$$

One could imagine applying at impulse of a different strength at every time point such that the pattern of impulses resembles a continuous input function. If the impulses form a continuous train then we can describe the resulting response using an integral:

$$y(t) = \int_0^\infty x(\tau)h(t-\tau)d\tau$$

where $x(\tau)$ is the function we are approximating by impulses. This integral is called the **convolution integral** and is often represented using the notation:

$$y(t) = x(t) * h(t)$$

Convolution means "a twisting together" and describes the result of applying an input signal to a system.

In the Laplace domain we know that:

$$Y(s) = X(s)H(s)$$

or

$$y(t) = \mathcal{L}^{-1}[X(s)H(s)]$$

from which we conclude that $X(s)H(s)$ is equivalent to taking the convolution in the laplace domain, that is

$$x(t) * h(t) = \mathcal{L}^{-1}[X(s)H(s)]$$

This result means that a convolution can be computed by multiplying the Laplace transforms together and then taking the inverse transform if the time domain response is required. This is simpler than trying to do the convolution integral in the time domain itself.

To determine the response of a system to an arbitrary, but continuous input function, it is much easier to work in the Laplace domain.