336: Systems and Control
State Space Examples (v1.0)

Contents

1 Example of State Space Models

1.1 Mechanical Model

Consider the following simple mechanical system:

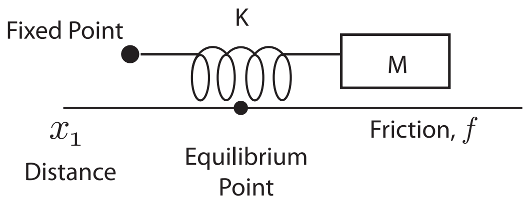

Figure 1: A mass, M, is attached to spring (K) which in turn is attached to a fixed point. The mass rests on a surface with friction, f.

Let us designate the distance moved relative to the equilibrium point as x_1. If the system moves to the left of the equilibrium point we assume $x_1 < 0$ otherwise we assume $x_1 > 0$. At equilibrium however, $x_1 = 0$. If we apply an impulse, $\delta(t)$, to the mass, it will begin to oscillate as it attempts to return to the equilibrium point. If we include a frictional force on the table then the movement of the mass will slow down until it reset again at the equilibrium point. We will assume that the velocity of the mass is given by x_2. Since x_2 is the velocity, the acceleration will in turn be given by the rate of change of the velocity:

$$a = \frac{dx_2}{dt}$$

The aim is to write down two differential equations, one for the velocity and another for the acceleration. To do this we must consider all the forces that contribute to the dynamics in this system. There are three forces involved, the force we apply when we displace the mass, the force that is exerted by the spring and the force exerted by the friction on the table. The total forces on the mass is therefore:

$$F_t = F_{impulse} + F_{spring} + F_{friction}$$

First consider the spring. Hooke's law states that the displacement experienced by a spring from it's equilibrium point is propositional to the applied force, F. Moreover because the force exerted by the spring opposes the direction of the displacement we also include a negative sign in the relationship. That is:

$$x = -\frac{1}{k}F$$

where k is called the spring constant. In the literature it is more often to find the following form:

$$F = -kx$$

The next force to consider is the frictional force. Friction always opposes motion and studies indicate that the frictional force is proportional to the rate at which the mass moves, in other words:

$$F = -f\frac{dv}{dt} = -f\frac{dx_1}{dt} = -fx_2$$

Again the force relationship is negative because as already stated, friction opposes movement.

The final force to consider is the displacement we apply to the mass at $t = 0$. This will be an impluse such that a $t < 0$ the system is at equilibrium but at $t = 0$ we apply an impulse, $\delta(t)$ that causes the system to respond. We can therefore write that the total force on the mass is:

$$F_{total} = \delta(t) - kx_1 - fx_2$$

From Newton's second law we know that the force on a mass is equal to the acceleration times the mass of the object. That is:

$$\frac{dx_2}{dt}M = \delta(t) - kx_1 - fx_2$$

or

$$\frac{dx_2}{dt} = \frac{\delta(t)}{M} - \frac{k}{M}x_1 - \frac{f}{M}x_2$$

This equation combined with the velocity equation gives us the complete system of equations for this system:

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{\delta(t)}{M} - \frac{k}{M}x_1 - \frac{f}{M}x_2$$

We can rewrite these equations in the state space representation as :

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{f}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} \left[\delta(t)\right]$$

Note that the impulse function is separated out from the rest of the model.

1.2 Electrical Model

Consider the electrical circuit shown in Figure 2:

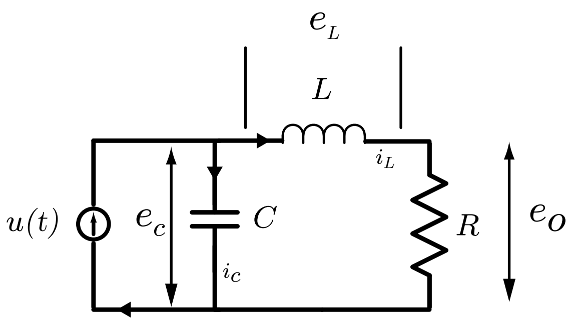

Figure 2: Electrical circuit comprising a fixed current course, $u(t)$ together with a capacitor, inductor and resistor. i is the current and e the voltage.

Let us note the following basic relationships:

$$e_o = Ri_L$$

$$i_c = C\frac{de_c}{dt}$$

$$e_L = L\frac{di_L}{dt}$$

$$e_c = e_L + e_o$$

$$u(t) = i_c + i_L$$

The circuit can be described by two differential equations, one describing the rate of change of the voltage across the capacitor and another that describes the rate of change of current through the inductor. From the capacitance law we can write:

$$\frac{de_c}{dt} = \frac{i_c}{C}$$

but since $i_c = u(t) - i_L$:

$$\frac{de_c}{dt} = \frac{u(t)}{C} - \frac{i_L}{C}$$

This provides one of the equations we need.

Given that $e_c = e_L + e_o$ the inductor law can be rewritten as:

$$L\frac{di_L}{dt} = e_c - e_o = e_c - Ri_L$$

or

$$\frac{di_L}{dt} = \frac{e_c}{L} - \frac{R_i}{L}i_L$$

which yields the second equation.

If we designate the capacitor voltage by x_1 and the inductor current by x_2 then we can rewrite these equations in the state space representation as:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{R}{L} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} [u]$$

The step input function to the system, $u(t)$ is separated out.

1.3 Pharmokinetic Model

Consider the pharmokinetic system shown in Figure 3:

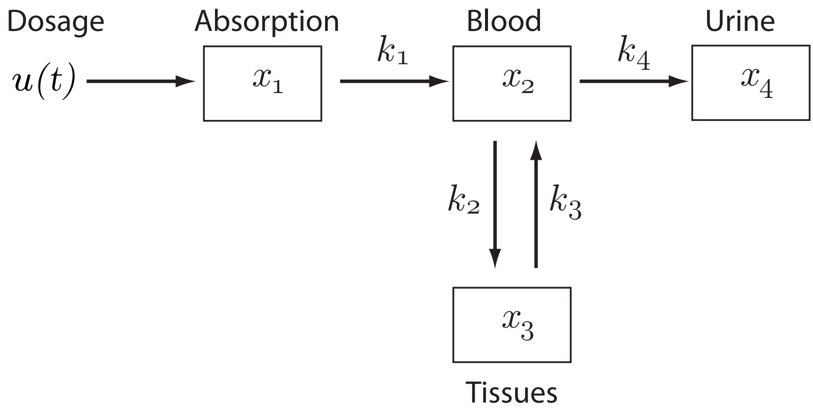

Figure 3: Pharmokinetic Model

We will assume first-order kinetics for each of the processes. The dosage will be assumed to be a step response so that:

$$\frac{dx_1}{dt} = u(t) - k_1 x_1$$

The remaining equations can be written in a similar fashion:

$$\frac{dx_2}{dt} = k_1 x_1 - k_2 x_2 - k_4 x_2 + k_3 x_3$$

$$\frac{dx_3}{dt} = k_2 x_2 - k_3 x_3$$

$$\frac{dx_4}{dt} = k_4 x_2$$

We can write the model in state space form as:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 & 0 \\ k_1 & -(k_2 + k_4) & k_3 & 0 \\ 0 & k_2 & -k_3 & 0 \\ 0 & 0 & k_4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

1.4 Biochemical Model

Consider the simple biochemical pathway below:

$$X_0 \xrightarrow{v_1} S_1 \xrightarrow{v_2} S_3 \xrightarrow{v_3} X_1$$

where we assume that X_o and X_1 are fixed species. Assume that the rate laws are given by:

$$v_1 = E_1(X_o k_1 - S_1 k_2)$$

$$v_2 = E_2(S_1 k_3 - S_2 k_4)$$

$$v_3 = E_3 k_5 S_2$$

where E_i is the concentration of enzyme i in reaction i. We could use nonlinear Michaelis-Menten rate laws but we can make the algebra simpler by assuming simple linear rate laws.

The differential equations for S_1, and S_2 are:

$$\frac{dS_1}{dt} = v_1 - v_2$$

$$\frac{dS_2}{dt} = v_2 - v_3$$

We will consider three parameters that serve as inputs, E_1, E_2 and E_3. If you have any problem formulating the state space representation even for linear systems one can use the linearization formula to derive the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \mathbf{x}_o}$$

For the system above, $\boldsymbol{A}$ is then easily derived as:

$$\boldsymbol{A} = \begin{bmatrix} -E_1 k_2 & k_4 E_2 \\ E_2 k_3 & -(E_2 k_4 + E_3 k_5) \end{bmatrix}$$

For $\boldsymbol{B}$:

$$\boldsymbol{B} = \frac{\partial \boldsymbol{f}}{\partial \mathbf{u}_o}$$

For the system above, $\boldsymbol{B}$ is derived ny assuming E_1, E_2 and E_3 are the parameters:

$$\boldsymbol{B} = \begin{bmatrix} k_1 X_o - k_2 S_1 & -(k_3 S_1 + k_3 S_2) & 0 \\ 0 & k_2 S_1 - k_4 S_2 & -k_5 S_2 \end{bmatrix}$$

Note that the $\boldsymbol{B}$ matrix is a 2 by 3 matrix corresponding to two state variables and three parameters.

The $\mathbf{u}_o$ vector is $[E_1 \;\; E_2 \;\; E_3]^T$. Fully written out the state space equation looks like:

$$\begin{bmatrix} \dfrac{dS_1}{dt} \\ \dfrac{dS_2}{dt} \end{bmatrix} = \begin{bmatrix} -E_1 k_2 & k_4 E_2 \\ E_2 k_3 & -(E_2 k_4 + E_3 k_5) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

$$+ \begin{bmatrix} k_1 X_o - k_2 S_1 & -(k_3 S_1 + k_3 S_2) & 0 \\ 0 & k_2 S_1 - k_4 S_2 & -k_5 S_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix}$$

One important point to bear in mind about the above equation. Even though matrix $\boldsymbol{B}$ matrix appears to be a function of S_1 and S_2, this is not true. The values for S_1 and S_2 must be taken from their values at the operating point (steady state). As a result, the $\boldsymbol{B}$ matrix contains only constant values.