336: Systems and Control

Solutions to $d\mathbf{x}/dt = A\mathbf{x}$

(v1.1)

Contents

1 Homogeneous Solutions

In the chapter on first-order systems it was found that the solution could be split into free and forced solutions. The free solution referred to the system where were no inputs and non-zero initial conditions. The free solution reflected the relaxation properties of the unforced system. In terms of the state space representaton, the free solution refers to the solution of the system:

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{A}\mathbf{x}$$

In order to discuss the solution to this system of differential equations we must first recall some properties of exponentials. The exponential function, e^{at} can be written using the well known series:

$$e^{at} = 1 + ta + \frac{t^2a^2}{2!} + \frac{t^3a^3}{3!} + \ldots$$

For the simple differential equation:

$$\frac{dx}{dt} = ax$$

the solution is known to be $x(t) = e^{at}x(0)$. Using the exponential series we can also write the solution in the form:

$$x(t) = \left(1 + ta + \frac{t^2a^2}{2!} + \frac{t^3a^3}{3!} + \ldots\right)x(0) \tag{1}$$

By analogy we might propose that for the system of equations:

$$\frac{d\mathbf{x}}{dt} = \boldsymbol{A}\mathbf{x}$$

the solution might look like:

$$\mathbf{x}(t) = \left(\boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \frac{\boldsymbol{A}^3 t^3}{3!} + \ldots \right) \mathbf{x}(0) \qquad (2)$$

We can check if this is indeed the solution by differentiating the proposed solution and see if we recover the system of differential equations. Differentiating equation 2 yields:

$$\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \left(\mathbf{0} + \boldsymbol{A} + \boldsymbol{A}^2 t + \frac{\boldsymbol{A}^3 t^3}{2!} + \ldots \right) \mathbf{x}(0) \\
&= \boldsymbol{A} \left(\boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \ldots \right) \mathbf{x}(0) \\
&= \boldsymbol{A}\mathbf{x}(t)
\end{aligned}$$

That is, equation 2 is the solution to $d\mathbf{x}/dt = \boldsymbol{A}\mathbf{x}$. By analogy, the matrix series in equation 2 is referred to as the **matrix exponential** as is represented by the curious expression:

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2 t^2 + \frac{1}{3!}\boldsymbol{A}^3 t^3 + \ldots$$

Although it may seem odd to raise e to the power of a matrix, the term, $e^{\boldsymbol{A}t}$ has properties very similar to a usual power. We can now write the solution to $d\mathbf{x}/dt = \boldsymbol{A}\mathbf{x}$ is a very compact way, namely:

$$\mathbf{x}(t) = e^{\boldsymbol{A}t}\mathbf{x}_o$$

Since the Laplace transform of $\boldsymbol{A}\mathbf{x}$ is $(s\boldsymbol{I} - \boldsymbol{A})^{-1})$ we can also state that:

$$\mathcal{L}^{-1}((s\boldsymbol{I} - \boldsymbol{A})^{-1}) = e^{t\boldsymbol{A}}$$

Example 1

Find the solution to the following system of differential equations assuming initial conditions, $x_1(0) = 2, x_2(0) = 3$:

$$\frac{dx_1}{dt} = -2x_1$$

$$\frac{dx_2}{dt} = x_1 - x_2$$

From the equations the $\boldsymbol{A}$ matrix is:

$$\begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$$

Using equation 2 we see can write:

$$e^{\boldsymbol{A}t} = \left(\boldsymbol{I} + \boldsymbol{A}t + \frac{\boldsymbol{A}^2 t^2}{2!} + \frac{\boldsymbol{A}^3 t^3}{3!} + \dots \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} t + \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} -8 & 0 \\ 7 & -1 \end{bmatrix} \frac{t^3}{3!} + \dots$$

$$= \begin{bmatrix} 1 - 2t + \frac{4t^2}{2!} - \frac{8t^3}{3!} + \dots & 0 \\ 0 + t - \frac{3t^2}{2!} + \frac{7t^3}{3!} + \dots & 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \end{bmatrix}$$

The elements in the final matrix can be seen as series representations for exponentials. In particular, the first element is e^{-2t} and the forth element e^{-t}. The third element is a bit more difficult to spot but can easily be shown to be $e^{-t} - e^{-2t}$. Therefore the solution to the original set of differential equations that incorporates the initial conditions is:

$$
\begin{aligned}
x_1(t) &= 2e^{-2t} \\
x_2(t) &= 2\left(e^{-t} - e^{-2t}\right) + 3e^{-t} \\
&= 5e^{-t} - 2e^{-2t}
\end{aligned}
$$

The approach used in this example is not that simple because it involves spotting the appropriate patterns in the series solution. In practice other more straight forward approaches are used, one of which is described in the Appendix.

Example 1 shows the solution to a simple set of linear differential equations without any forcing terms. In state space these solutions represent the free response. The solution to the equations was a weighted sum of exponentials and is a result that can be generalized such that the solution to a set of linear time independent differential equations without forcing terms is given by:

$$
x_j(t) = \sum_{k=1}^{n} \beta_{jk} e^{\lambda_k t}
$$

The solution involves the sum of weighted (β_{jk}), exponentials, $e^{\lambda_k t}$. The exponents of the exponentials turn out to be the eigenvalues of the matrix, $\boldsymbol{A}$, that is the Jacobian matrix. If the eigenvalues are negative then the exponents decay (stable) whereas if they are positive the exponents grow (unstable). We can therefore determine the stability properties of a given model by computing the eigenvalues of the Jacobian matrix and looking for any positive eigenvalues.

Sometimes the series in equation 2 will admit solutions that look like:

$$
t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots
$$

For those familiar with trigonometric series, this series will be recognized as representing $\sin(t)$. In other words the solutions can admit sinusoidal elements. In terms of the generalized exponential solution, this corresponds to complex λ terms in the exponents.

2 Stability from the Transfer Function

A system can be said to be stable if small perturbations in the state variables around the steady state decay back to the steady state. Since the linearized system describes the rate of change of a disturbance, all we need do is look at the eigenvalues of in the time domain solution. If all the eigenvalues are negative this will mean that the perturbations will decay to zero.

Can we say something about stability from the Transfer function?

Consider the simple first-order system:

$$H(s) = \frac{1}{s+k}$$

We know that the time-domain solution to this is the free response that describes the evolution of a state variable in the absence of an input. We also know that the time domain solution is:

$$y(t) = y(0)e^{-kt}$$

This clearly shows that an initial condition (or disturbance around the steady state) will decay to zero. This system must be stable. What if we look at the slightly different transfer function:

$$H(s) = \frac{1}{s-k}$$

All we have done is changed the sign of one term in the denominator (the characteristic equation). The solution to this system is:

$$y(t) = y(0)e^{kt}$$

That is the system grows exponentially, not a very stable system. Information on the stability of a systems lies in the denominator of the Laplace transform. In general we could write a transfer function in the following way:

$$H(s) = \frac{F(s)}{(s + a_1)(s + a_2) \dots (s + a_m)}$$

Provided that all the a values are positive, the system will be stable. This corresponds to all the **roots** of the denominator being **negative**.

Looking at the roots of the characteristic equation or the eigenvalues of the Jacobian matrix ($\boldsymbol{A}$) is equivalent.

In control theory terminology, the roots of the characteristics equation are called the **poles** and the roots of the numerator are called the **zeros**.

For higher order systems finding the roots of the characteristic equation is not easy, therefore a more heuristic method was devised called the **Routh Hurwitz** Stability Method that can be used to determine the stability without having to explicitly obtain the roots (See Network Dynamics Sauro, H M. (2009) Computational Systems Biology Series: Methods in Molecular Biology , Vol. 541 McDermott, J.; Samudrala, R.; Bumgarner, R.; Montgomery, K.; Ireton, R. (Eds.) ISBN: 978-1-58829-905-5 for simple examples).