336: Systems and Control
Common Input Signals v1.3

Contents

1 Signals

There are a number of common ways in which engineers perturb systems. Such perturbations are often referred to as input functions, forcing functions, or a driving function.

In this chapter we are going to look at a number of them, including the **step**, **impulse**, the **ramp** and the **sinusoidal** input. Along the way we will also introduce **shifting** or time delays in signals. In the following text, t refers to time.

1.1 Unit Step Function

The unit step function, $u(t)$, is described as a function in time that has a value of zero **before** $t = 0$ but from $t \geq 0$, it has a constant value of one (Figure 1). We can define the unit step function in the following way:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

The Laplace transform for the unit step function is:

$$\mathcal{L}[u(t)] = \frac{1}{s}$$

For steps other than unity, such as a step size of T we can write the step function as:

$$u_T(t) = T\, u(t)$$

Using the linearity rule, the Laplace transform is:

$$\mathcal{L}[u_T(t)] = \frac{T}{s}$$

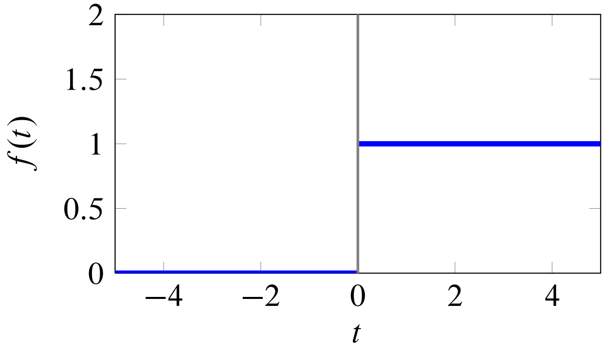

Figure 1: Unit Step Function, $u(t)$

1.2 Delayed Signals or Shifting

It is often desirable to delay a signal. For example rather than have the step response occur at $t = 0$, we might want the step to occur at some later time. Delayed responses are so common that a particular notation is used to indicate a delay. A delay is also called a **shift**. For example, Figure 2 show a step response delayed until $t = 2$. To indicate such delays, we use the notation:

$$u(t - a)$$

where a is the amount of delay. Logically this expression makes sense. When $t = a$, $t - a = 0$. that is when time reaches a, $f(t - a) = f(0)$ that is the value it has a time zero. Similarly if $t < a$, then the value of $f()$ corresponds to a time before $t = 0$, that is zero. The function therefore moves forward by a time units.

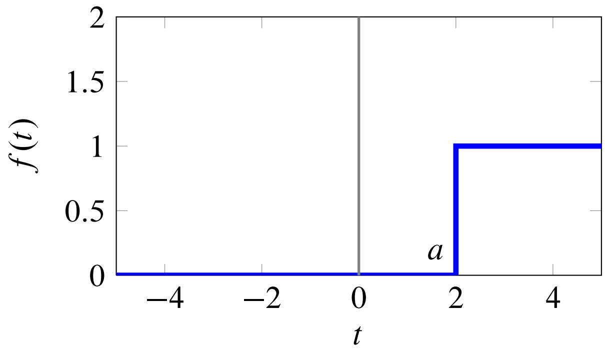

Figure 2: Delayed Unit Step Function, $u(t-2)$

1.3 Pulses

By combining step functions and delays we can define pulse signals. For example lets say we want a pulse to start at $t = a$ and to stop at a later time $t = b$. Such a situation can be represented as follows:

$$p(a,b) = u(t-a) - u(t-b)$$

That is, at time a the unit step function rises to 1.0. At this point $u(t-b)$ is still zero. At $t = b$, $u(t-b)$ goes to one but it subtracts this value from $u(t-a)$ thereby reducing the signal back to zero. Figure 3 illustrates such a response.

1.4 Ramp Signal

The ramp is a function of time that rises at a constant rate. It can be conveniently defined in terms of the unit step response as:

$$r(t) = t\ u(t)$$

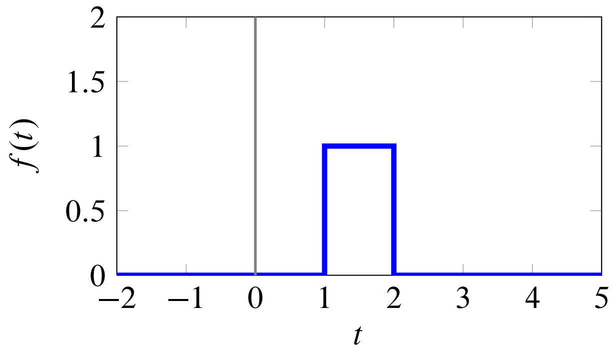

Figure 3: Unit Pulse Signal, $p(a, b) = p(1, 2)$

or more simply:

$$r(t) = at$$

The advantage of the former description is that it is easy to define a ramp that is shifted in time. Since the unit step function is zero up to $t = 0$, the ramp is also zero. Only when $t > 0$ does the ramp begin to rise and then at a rate related to the size of the step signal, Figure 4. Like the step function, the ramp can also be delayed, for example the following expression:

$$r(t - a) = t\ u(t - a)$$

means that the ramp only starts to rise at $t = a$. The slope of the ramp can be adjusted by multiplying the function by a constant, R, that is: $r_R(t - a) = R\ t\ u(t - a)$.

More complicated ramps can be constructed by summing additional terms. For example a ramp that rises than stops (Figure 5) can be defined using the expression:

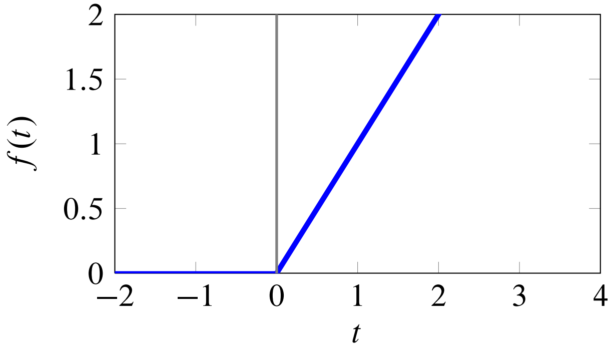

Figure 4: Unit Ramp Function, $r(t)$

$$Rtu(t) - R(t-a)u(t-a)$$

The first term describes a ramp starting at time, t with slope R. At time a, the same slope is now subtracted from the original rising slope which gives us a horizontal line. The term $u(t-a)$ ensures that the negative term is only non-zero at time a.

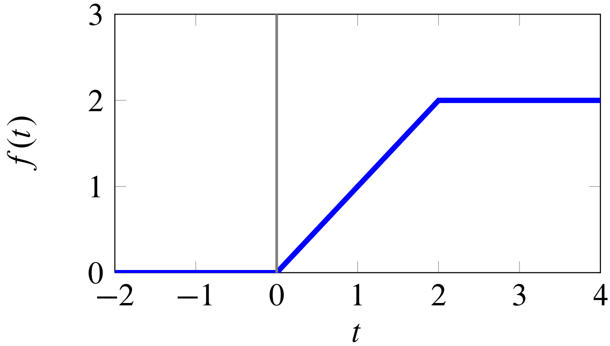

Figure 5: Unit Ramp Function that stops at time 2 with slope 1, $tu(t) - (t-2)u(t-2)$

The Laplace transform for the ramp, $r(t)$ is:

$$\mathcal{L}[r(t)] = \frac{1}{s^2}$$

1.5 Impulse Signal

The impulse (also known as the Dirac delta function, $\delta(t)$) is probably the most important signals and at the same times one of the odder "functions' to consider. The perfect impulse is a signal that lasts an infinitesimally small amount of time but whose magnitude appears to be infinite. Such a thing doesn't exist in the physical world and yet is extremely important in control theory and in practice the impulse can only be realized approximately. Mathematically we can say that the impulse has zero value everywhere except at $t = 0$ where its value is very large. The impulse is usually defined in the following way:

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

The second expression states that the area of an impulse is one. This is because the width of the pulse is T (Figure 6) while the height is $1/T$ so that as T is reduced to a smaller and smaller value the area remains fixed.

We can also scale an impulse (effectively scaling its area) such that:

$$\int_{-\infty}^{\infty} a\delta(t) dt = a$$

Like other signals the impulse can be delayed so that we can specify that a pulse should occur at a delayed time a using the notation:

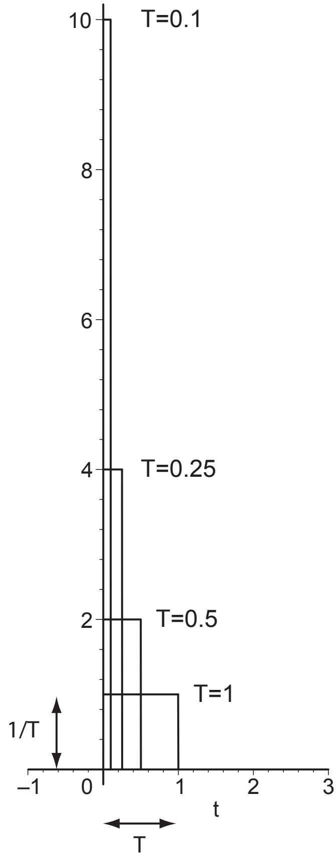

Figure 6: Pulses converging to an impulse while maintaining the area fixed at 1.0: $T \times 1/T$

$$\delta(t - a)$$

Alternatively we can also write the delay using the unit step function:

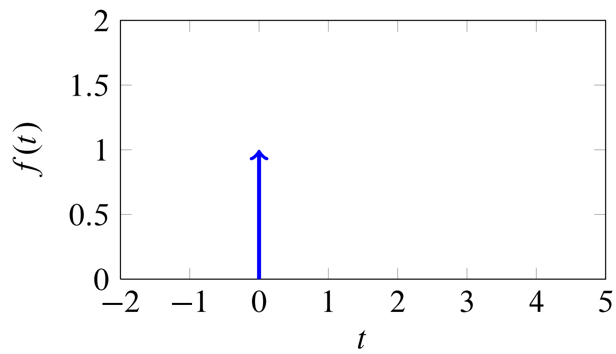

Figure 7: Impulse Signal, $\delta(t)$

$$u(t-a)\ \delta(t)$$

A series if impulses can be represented as a sum of impulses are different times:

$$u(t-a)\delta(t)+u(t-b)\delta(t)+\dots$$

Another useful property of the impulse is its ability to pick out a value from a given function, called *sifting*. Consider a continuous function, $f(t)$ that we multiply by $\delta(t)$. This product will be zero except for the point marked by the impulse. At that point, $f(t) = f(0)$, that is:

$$f(t)\delta(t) = f(0)\delta(t)$$

This can be generalized to

$$f(t)\delta(t-a) = f(a)\delta(t-a)$$

On its own, the above relation is not very useful but becomes much more interesting when we integrate the right-hand term:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)dt = \int_{-\infty}^{\infty} f(a)\,\delta(t-a)dt$$
$$= f(a)\int_{-\infty}^{\infty} \delta(t-a)dt = f(a)$$

In summary:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{1}$$

One final property worth noting is the relationship between an impulse and the step function. If we think about it the step function is itself a little odd because the derivative as the step is essentially infinite. Since the point at which the step occurs is infinitesimally small, the derivative of the step is reminiscent of an impulse. We can state that:

$$\frac{du(t)}{dt} = \delta(t)$$

Of particular interested to control theorists is that the Laplace transform of the impulse function.

$$\mathcal{L}\left[\delta(t-a)\right] = \int_{0}^{\infty} e^{-st}\delta(t-a)dt$$

This equation looks like the shifting property (1) which picks out a value. Using this property we therefore pick out the value e^{-st} at a, that is:

$$\mathcal{L}\left[\delta(t-a)\right] = \int_{0}^{\infty} e^{-sa}dt = e^{-sa}$$

If no shift occurs, that is $a = 0$, then the Laplace transform becomes 1, that is:

$$\mathcal{L}[\delta(t)] = 1$$

1.6 Sinusoidal

The sine function is another signal that finds common using in engineering. The sinusoidal signal is the most fundamental periodic signal. Any other periodic signal can be constructed from a summation of sinusoidal signals. We can describe a sinusoidal signal by an equation of the general form:

$$y(t) = A \sin(\omega t + \theta)$$

This equation describes how an output, y, varies in time, t. The equation has three terms, the amplitude (A), the angular frequency (ω) and the phase (θ). We will describe the phase in more detail below but essentially it indicates how delayed or advanced and periodic signal may be. The angular frequency is the rate of the periodic signal and is expressed in radians per second. A sine wave traverses a full cycle (peak to peak) in 2π radians (circumference of a circle) so that the number of complete cycles traversed in one second is then $\omega/2\pi$. This is termed the frequency, f, (cycles sec^{-1}) of a sine wave and has units of Hertz. The angular frequency is then conveniently expressed as $\omega = 2\pi f$.

The amplitude is the extent to which the periodic function changes in the y direction, that is the maximum height of the curve from the origin (Figure 9). The horizontal distance peak to peak is referred to as the period, T and is usually expressed in seconds. The inverse of the period, $1/T$ sec^{-1} is equal to the frequency, f.

Figure 11 shows a typical plot of the sinusoidal function, $y(t) =$

$A\sin(\omega t+\theta)$, where the amplitude is set to one, the frequency to 2 cycles per second and the phase to zero.

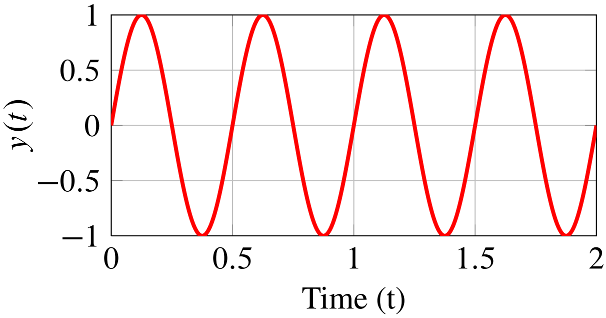

Figure 8: $y(t) = A\sin(\omega t + \theta)$ where $A = 1$, $f = 2$ Hz so that $\omega = 4\pi, \theta = 0$.

The left panel in Figure 9 shows the affect of varying the amplitude and right panel two signals of different frequency.

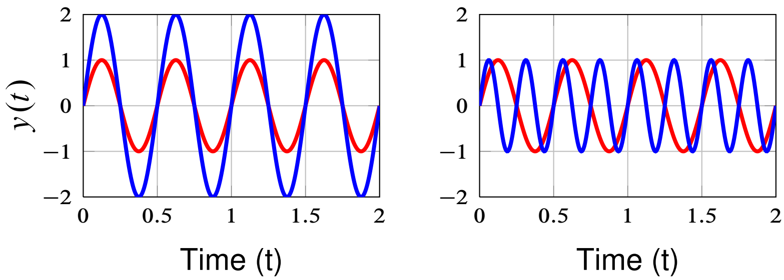

Figure 9: Left Panel: Amplitude change $y(t) = A\sin(\omega t + \theta)$ where $A = 2$, $f = 2$ Hz, $\theta = 0$ Right Panel: Frequency Change $y(t) = A\sin(\omega t + \theta)$ where $A = 1$, $f = 4$ Hz, $\theta = 0$

The phase start for a sinusoidal signals need not be zero. The two sine waves shows in Figure 10 have the same frequency and ampli-

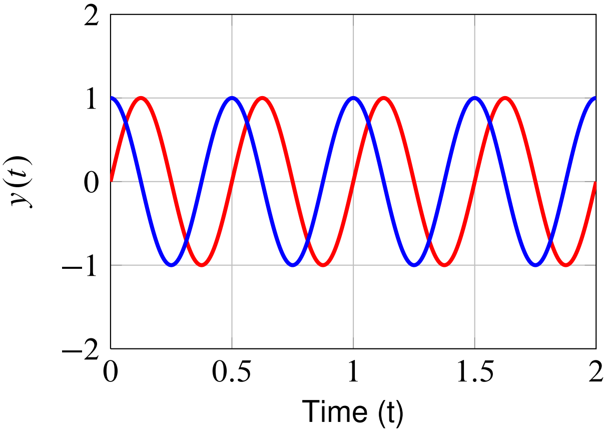

Figure 10: Phase change: $y(t) = A\sin(\omega t + \theta)$ where $A = 1, f = 2$ Hz, $\theta = 90°$. The red curve is shifted 90° to the right relative to the blue curve.

tude but one of them is shifted to the right by 90 degrees (or $\pi/2$ radians), that is phase shifted.

The sine signal can be delayed by multiplying it by a delayed unit step response, for example:

$$u(t-a)\ \sin(\omega t)$$

The Laplace transform of the sine function is given by:

$$\mathcal{L}[A\sin(\omega t)] = \frac{A\omega}{s^2+\omega^2}$$

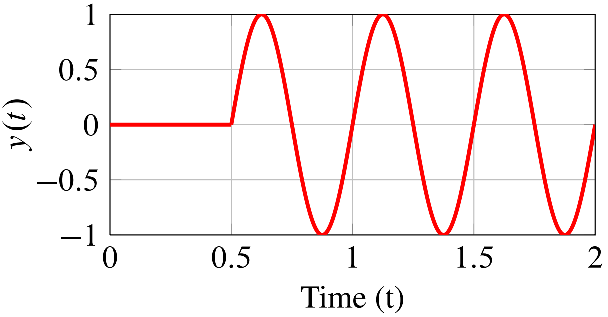

Figure 11: Delayed Sine Function, $u(t - 0.5)\ \sin(4\pi t)$