

336: Systems and Control
Introduction to Laplace Transforms
v1.21

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1 Laplace Transforms for Users

Solving linear differential equations by hand can be tedious, although with modern symbol software such as Mathematica it can now be done relatively easily. However a classical approach to solving linear differential equations and which has considerable other advantages, is the Laplace transform. Here we will describe the use of the Laplace transform in solving linear differential equations. The method is analogous to using logarithms to make multiplication easier. Here we take log of the two numbers we wish to multiply, add the resulting values, then take the antilog to obtain the product. The argument is that simple addition is much easier to carry out compared to multiplication. The same strategy can be used in solving linear differential equations. In this case we take the Laplace transform of the differential equations which turns them into algebraic equations, we solve the algebraic equations for the unknown variables and then take the inverse Laplace transform to recover the solution.

It was the 19th century French mathematician and astronomer, Pierre Simon de Laplace who developed the Laplace transform as a means for solving differential equations. The Laplace transform is defined as follows. If a function of time, $f(t)$ is defined for all values of $t \geq 0$, and if s is a complex variable, $s = \sigma + j\omega$ then the Laplace transform, $F(s)$ of $f(t)$ is given by:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

In this form the Laplace transform is also referred to as the unilateral Laplace transform to distinguish it from the bilateral form where the integration goes from minus infinity to plus infinity.

In order for the Laplace transform method to work we must make sure that for some finite value of s the integral actually converges.

This is possible if either $f(t)$ decays to zero at infinite t , or increases at a slower rate than the rate of decay of e^{-st} . If this is the case then $F(s)$ can be defined and is called the transform of $f(t)$, often symbolized using $\mathcal{L}()$. Note that integral starts at $t = 0$ and is therefore only defined for positive t .

Step Function Let try a very simple function and compute its Laplace transform. The simplest function is $f(t) = 1$ for $t > 0$. That is at $t \leq 0$ $f(t) = 0$. This represents a step function, often symbolized using $u(t)$ (Figure 1) of size one at $t = 0$. Since $f(t) = 1$, the Laplace transform will look like:

$$F(s) = \int_0^{\infty} e^{-st} 1 dt = \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t=\infty} = \frac{1}{s}$$

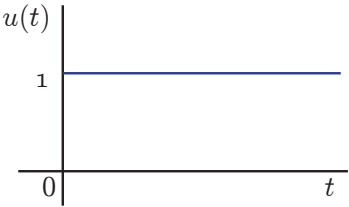


Figure 1: Step Function, $u(t)$

Exponential Function Derive the Laplace transform of the exponential function:

$$f(t) = e^{-kt}$$

Inserting $f(t)$ into the Laplace transform:

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-kt} e^{-st} dt$$

$f(t)$	$\mathcal{L}[f(t)]$
a	$\frac{a}{s}$
$f(t)$	$F(s)$
ae^{-kt}	$\frac{a}{s+k}$
$\frac{a}{b}(1 - e^{-bt})$	$\frac{a}{s(s+b)}$
$\frac{df(t)}{dt}$	$sF(s) - f(0)$
$af_1(t) + bf_2(t)$	$aF_1(s) + bF_2(s)$

Table 1: Brief Laplace Transform Table

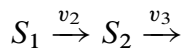
$$\mathcal{L}[f(t)] = \left[\frac{1}{s+k} e^{-(s+k)t} dt \right]_0^\infty$$

and applying the integral limits we obtain:

$$\mathcal{L}[f(t)] = \frac{1}{s+k}$$

Most of the time there is no need to derive the transform directly because there are published tables that list common functions and the corresponding Laplace transform, Table 1.

Let us apply the Laplace transform method to a real problem. Consider the three step pathway:



S_2 drains into the environment. The system therefore has two state variables, S_1 and S_2 . The system of differential equations for this model is:

$$\begin{aligned}\frac{S_1}{dt} &= -k_1 S_1 \\ \frac{S_2}{dt} &= k_1 S_1 - k_2 S_2\end{aligned}$$

For the sake of consistent notation let us rename the variables to $f_1(t)$ and $f_2(t)$ such that:

$$\begin{aligned}\frac{df_1(t)}{dt} &= -k_1 f_1(t) \\ \frac{df_2(t)}{dt} &= k_1 f_1(t) - k_2 f_2(t)\end{aligned}$$

Applying the Laplace rules from Table 1, we obtain:

$$\begin{aligned}sF_1(s) - f_1(0) &= -k_1 F_1(s) \\ sF_2(s) - f_2(0) &= k_1 F_1(s) - k_2 F_2(s)\end{aligned}$$

The above set of equations represent a set of linear algebraic equations in $F_1(s)$ and $F_2(s)$. $f_1(0)$ and $f_2(0)$ represent the initial conditions for S_1 and S_2 at time zero. If we assume that $S_1 = a$ and $S_2 = 0$ at time zero, then:

$$\begin{aligned}sF_1(s) - a &= -k_1 F_1(s) \\ sF_2(s) &= k_1 F_1(s) - k_2 F_2(s)\end{aligned}$$

Solving for $F_1(s)$ and $F_2(s)$ yields:

$$F_1(s) = \frac{a}{s + k_1}$$
$$F_2(s) = \frac{k_1 a}{(s + k_1)(s + k_2)}$$

Now that we have solved for $F_1(s)$ and $F_2(s)$ we must apply the inverse Laplace transform to recover the time domain solution. The inverse transform can be obtained by using the Laplace transform table backwards. However we do not have entries in the table that resemble the right-hand side of $F_2(s)$. In order to proceed we must split the equation into simpler terms. The hope is that the simpler terms will be found in a table of Laplace transforms. To do this we can use partial fractions. Let us first however find the inverse transform of $F_1(s)$, this is straight forward because the term $a/(s + k_1)$ is the 3rd entry in the table, that is:

$$S_1 = a e^{-k_1 r}$$

For the second equation we will use partial fractions.

$$F_2(s) = \frac{k_1 a}{(s + k_1)(s + k_2)} \quad (1)$$

Let us separate $F_2(s)$ as follows:

$$\frac{k_1 a}{(s + k_1)(s + k_2)} = \frac{A}{s + k_1} + \frac{B}{s + k_2}$$

Multiply both side by the denominator of the left side to yield:

$$k_1 a = A(s + k_2) + B(s + k_1)$$

Multiply out the brackets and collect like ‘s’ terms:

$$k_1a = As + Bs + Ak_2 + Bk_1$$

$$k_1a = s(A + B) + Ak_2 + Bk_1$$

By matching terms on the left and right we can conclude that:

$$A + B = 0$$

$$k_1a = Ak_2 + Bk_1$$

From these two linear equations we can determine A and B :

$$B = -A = \frac{k_1a}{k_1 - k_2}$$

Inserting A and B yields:

$$F_2(s) = \frac{k_1a}{(k_1 - k_2)} \frac{1}{s + k_2} - \frac{k_1a}{(k_1 - k_2)} \frac{1}{s + k_1}$$

We can now use the reverse transform of $a/(s + k)$ to yield an equation for $S_2(t)$:

$$S_2(t) = \frac{k_1a}{(k_1 - k_2)} \left[e^{-k_2t} - e^{-k_1t} \right]$$

Recall that $a = S_1(0)$. The use of the inverse transform to derive differential equations is not so important today given the availability of symbolic applications such as Mathematica or the open source tools such as Sage (<http://www.sagemath.org/>) or wxMaxima (<http://andrejv.github.com/wxmaxima/> and Maxima <http://maxima.sourceforge.net/>). However, the ability to form the

transforms is still very important because the elimination of time and its replacement with s leads to a host of new analysis methods.

2 Properties of the Laplace Transform

2.1 Linearity and Constant Multiple

$$\mathcal{L}[af(t)] = a\mathcal{L}[f(t)]$$

For example, if:

$$\mathcal{L}\left[\frac{1}{b}(1 - e^{-bt})\right] = \frac{1}{s(s - b)}$$

then

$$\mathcal{L}\left[\frac{a}{b}(1 - e^{-bt})\right] = \frac{a}{s(s - b)}$$

Summation Rule:

$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)]$$

More generally:

$$\mathcal{L}[a_1f_1(t) + a_2f_2(t)] = a_1\mathcal{L}[f_1(t)] + a_2\mathcal{L}[f_2(t)]$$

2.2 Final-Value Theorem:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

This theorem is very useful for computing the behavior of the model at infinite time when the system might be at thermodynamic equilibrium or steady state. In the previous example the two transforms were:

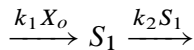
$$F_1(s) = \frac{a}{s + k_1}$$

$$F_2(s) = \frac{k_1 a}{(s + k_1)(s + k_2)}$$

If we apply the final-value theorem we find that both S_1 and $S_2 \rightarrow$ to zero. This is as expected, because all the mass will drain from the pathway leaving nothing in S_1 and S_2 ,

Question 1

Use Laplace transforms to find the solution to the following simple reaction system:



X_o is fixed constant concentration with rate constant k_1 and k_2 is the rate constant for the first-order degradation reaction.

Use the final-value theorem to determine the steady-state level of S_1 .

3 Nonlinear Systems

It is not possible to apply the Laplace transform method to nonlinear systems but we can do the next best thing and linearize the system.

3.1 State Space Representation

We can represent a general system as follows (Figure 2):

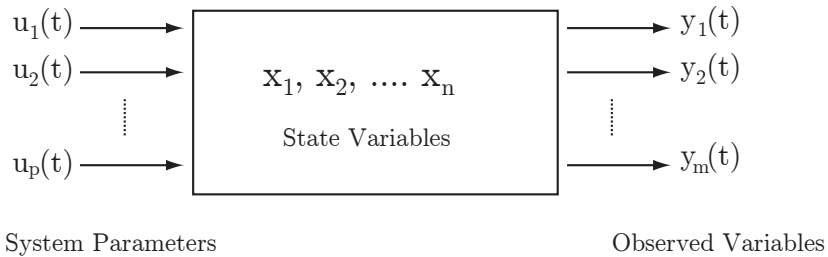


Figure 2: Systems Diagram with Inputs and Outputs.

This representation has three types of symbol: the state variables, x_i that represent the quantities that define the behavior of the system; the parameter inputs, u_i , which represent quantities that are fixed properties of the system that can be controlled by the observer; and the system output, y_i , which represent the actual observables. In simple systems the state variables are often the same as the observables but sometimes one can only indirectly access the state variable via other kinds of measured quantities (y_i).

To give a concrete example of a system, consider a simple genetic circuit that expresses green fluorescence protein in response to changes in an inducer molecule. Let the state variable of the system be the concentration of expressed protein, this would be a x_i type quantity in Figure 2. However we can't actually observe the protein directly, only as a fluorescent measurement in an instrument. The fluorescence we actually observe in the experiment is represented by a y_i variable in Figure 2. Finally, the inducer that we can apply to the system to change the level of gene expression is an external parameter, u_i .

In the following notation, lower case bold will be used to represent vectors and upper case bold will be used to represent matrices.

Let us represent the vector of state variables, \mathbf{x} , by:

$$\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$$

the vector of inputs, \mathbf{u} , by

$$\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_p(t)]^T$$

and the vector of output variables by:

$$\mathbf{y}(t) = [y_1(t), y_2(t), \dots, y_m(t)]^T.$$

In general the relationship between the state variables, x_i and the input parameters, u_i is nonlinear and can be written down as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$$

where \mathbf{x} and \mathbf{u} are vectors. Likewise the vector of observables, \mathbf{y} is related to the state variables and parameters by some nonlinear function \mathbf{g} :

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$$

\mathbf{x} , \mathbf{u} and \mathbf{y} are in general functions of time. For brevity, we will omit the (t) in the following discussion.

It is common to linearize these nonlinear system equations around the equilibrium or steady state point. If we let \mathbf{x}_o and \mathbf{u}_o be the values at steady state then by definition it must be the case that:

$$0 = \mathbf{f}(\mathbf{x}_o, \mathbf{u}_o)$$

Let us now make a small disturbance around \mathbf{u}_o such that:

$$\mathbf{u} = \mathbf{u}_o + \delta\mathbf{u}$$

and

$$\mathbf{x} = \mathbf{x}_o + \delta\mathbf{x}$$

We now expand $f(\mathbf{x}(t), \mathbf{u}(t))$ around \mathbf{x}_o and \mathbf{u}_o using a Taylor series to obtain:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}_o, \mathbf{u}_o) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}_o}(\mathbf{x} - \mathbf{x}_o) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}_o}(\mathbf{u} - \mathbf{u}_o) + \dots$$

We now define:

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}_o}$$

$$\mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}_o}$$

Since $\mathbf{x} = \delta\mathbf{x} + \mathbf{x}_o$ and \mathbf{x}_o is a constant:

$$\frac{d\mathbf{x}}{dt} = \frac{d\delta\mathbf{x}}{dt}$$

Finally, noting that $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_o$ and $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}_o$ we obtain:

$$\frac{d}{dt}\delta\mathbf{x} = \mathbf{A}\delta\mathbf{x} + \mathbf{B}\delta\mathbf{u}$$

This is a linear time independent equation that describes the rate of change of the perturbation in \mathbf{x} around the steady state as a result of perturbations in \mathbf{x} and/or \mathbf{u} .

The function \mathbf{g} can also be linearized to obtain:

$$\delta\mathbf{y} = \mathbf{C}\delta\mathbf{x} + \mathbf{D}\delta\mathbf{u}$$

The matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} are constant and **do not** depend on time or the state variables. In the control literature, these matrices are given the following labels:

- A** $n \times n$ State Matrix, or Jacobian
- B** $n \times p$ Control Matrix
- C** $m \times m$ Output Matrix
- D** $m \times p$ Feed-forward Matrix

If the **C** matrix is the identity matrix and the **D** matrix is zero, then the state variable vector is equal to the output vector, **y**. The linearized state equations are often represented in the literature by equation (2). For many systems the **D** matrix is empty. A block diagram representing of equations (2) is given in Figure 3 which shows why the **D** matrix is called the feed-forward matrix.

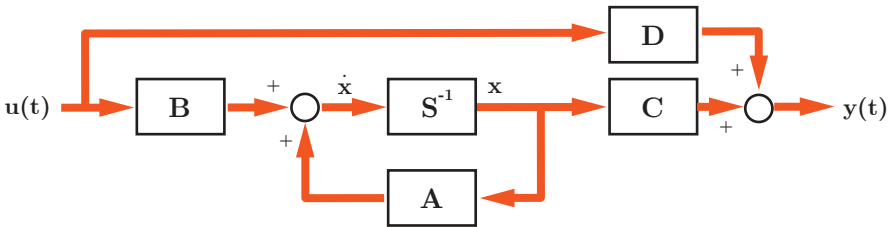


Figure 3: Block diagram representation of the state space formulation.

$$\frac{dx}{dt} = Ax(t) + Bu(t) \tag{2}$$

$$y(t) = Cx(t) + Du(t)$$

These equations are both linear and time-invariant and are called the **state space equations**.

Written out, the state space equation for \mathbf{x} would look like:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$$

Note that the \mathbf{A} matrix is always square.

3.2 Laplace Transform

Since the state equations are linear we can apply the Laplace transform to them, first we look at the $d\mathbf{x}/dt$ equation:

$$\mathcal{L}\left(\frac{d\mathbf{x}}{dt}\right) = \mathcal{L}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})$$

$$s\mathbf{X}(s) - \mathbf{x}_o = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

where \mathbf{x}_o is the vector of initial conditions. Collecting the term $\mathbf{X}(s)$ yields:

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}_o + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}(\mathbf{x}_o + \mathbf{B}\mathbf{U}(s))$$

If we assume that the initial conditions, \mathbf{x}_o are zero then

$$\mathbf{X}(s) = [(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]\mathbf{U}(s)$$

The term $(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ is called the **transfer function matrix**, sometimes indicated by $\mathbf{H}(s)$:

$$X(s) = H(s)U(s)$$

What we have done is split the system into two parts, $H(s)$ which is the intrinsic property of the system and $U(s)$ which is the perturbation we apply to the system. Both together give us the effect a perturbation has on the outputs of the system, $X(s)$.

If the C matrix is not the identity matrix and D is non-zero then we can further generalize to:

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) \quad (3)$$

Example 1

Derive the transfer function for the following system, where x_o is fixed:

$$x_o \xrightarrow{k_1 x_o} x \xrightarrow{k_2 x}$$

with ODE:

$$\frac{dx}{dt} = k_1 x_o - k_2 x$$

We can cast the ODE in the state space representation:

$$\frac{dx}{dt} = [-k_2]x + [k_1]x_o$$

From equation (3) we have

$$A = [-k_2]$$

$$B = [k_1]$$

Assuming $C = I$ and $D = \mathbf{0}$ we can use equation (3) to obtain:

$$Y(s) = (sI - A)^{-1}B U(s)$$

that is:

$$Y(s) = (s + k_2)^{-1} [k_1] U(s)$$

The transfer function is:

$$H(s) = \frac{k_1}{s + k_2}$$

4 Applications of the Laplace Transform

4.1 Solving Linear ODEs

4.2 Solving Steady State

4.3 Frequency Response

4.4 Systems Analysis

4.5 Impulse and Step Function Responses

4.6 Reverse Engineering a System