336: Systems and Control
First-Order Systems (v1.21)

Contents

1 First-Order Systems

In previous notes the following example was presented.

$$\boldsymbol{Y}(\mathbf{s}) = \left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right]\boldsymbol{U}(s) \tag{1}$$

Example 1

Derive the transfer function for the following system, where X_o is fixed:

$$x_o \xrightarrow{k_1 X_o} y \xrightarrow{k_2 x}$$

with ODE:

$$\frac{dy}{dt} = k_1 X_o - k_2 y$$

We can cast the ODE in the state space representation:

$$\frac{dy}{dt} = [-k_2]y + [k_1]X_o$$

From equation (1) we have

$$\boldsymbol{A} = [-k_2]$$

$$\boldsymbol{B} = [k_1]$$

Assuming $\boldsymbol{C} = \boldsymbol{I}$ and $\boldsymbol{D} = \mathbf{0}$ we can use equation (1) to obtain:

$$\boldsymbol{Y}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\ \boldsymbol{U}(s)$$

that is:

$$\boldsymbol{Y}(s) = (s + k_2)^{-1}\ [k_1]\ \boldsymbol{U}(s)$$

The transfer function is:

$$\boldsymbol{H}(s) = \frac{k_1}{s + k_2}$$

If we look at the denominator of the transfer function we see that the highest power that s is raised to is one. We therefore call this kind of system a **first-order system**. In these notes we will investigate first-order systems in more detail.

The term $U(s)$ is equal to X_o/s because $u(t) = X_o$ and the output equation becomes:

$$Y(s) = \frac{k_1}{(s + k_2)} \frac{X_o}{s}$$

The derivation assumed that the initial condition for the state variable y was zero. If we relax this restriction we will get instead the following general relation for the input/output relationship which now include the initial condition, $\mathbf{y}_o$:

$$\boldsymbol{Y}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\mathbf{y}_o$$

or in terms of the transfer function, $\boldsymbol{H}(s)$:

$$\boldsymbol{Y}(s) = \boldsymbol{H}(s)\boldsymbol{U}(s) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\mathbf{y}_o$$

Applying this equation to Example 1 and taking the inverse Laplace yields:

$$Y(s) = \frac{k_1}{(s + k_2)} \frac{X_o}{s} + \frac{1}{(s + k_2)} y(0)$$

We can take the inverse transform of $Y(s)$ to obtain the time evolution of the system, that is:

$$y(t) = \frac{X_o k_1}{k_2} \left(1 - e^{-k_2 t}\right) + y(0) e^{-k_2 t} \quad (2)$$

Solution 2 is called the **total or complete solution** because it includes both the non-zero initial conditions and apply an input function (in this case a step function).

Equation 2 is actually quite interesting. We can surmise some generalities from this equation that apply to all LTI systems.

The first thing to note is that the equation is in two parts. The first part is given by:

$$y(0)e^{-k_2 t}$$

This is called the **free response** because it doesn't include the input step function and assumes that the initial conditions are not necessarily zero. It is the response of the system in the absence of any input. Figure 1 shows the response of the free system to a positive initial condition. The natural response of the free system is to decay to zero.

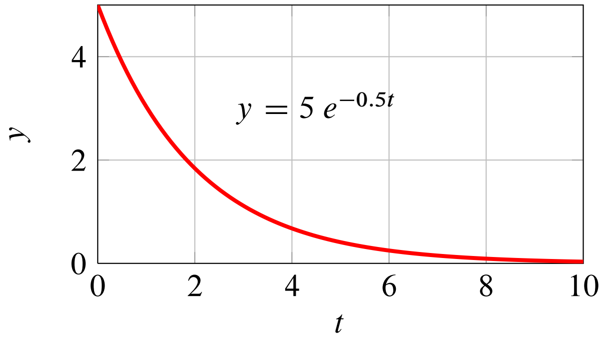

Figure 1: Free Response for a First-Order System

The second part of the total response is given by:

$$\frac{X_o k_1}{k_2}\left(1 - e^{-k_2 t}\right)$$

In this case the equation on its own assumes that the initial condition, $y(0)$ is zero but has an input step function with respect to X_o. This part of the equation is called the **forced response**. Figure 2 shows an example of a forced response on the first-order system.

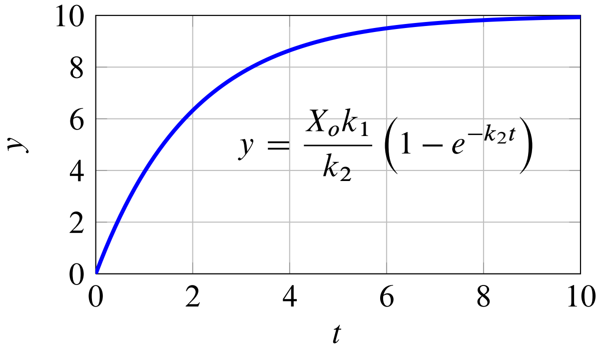

Figure 2: Free Response for a First-Order System, $v_o = 5, k_1 = 0.5$

The force response equation itself can also be divided into two parts:

$$\frac{X_o k_1}{k_2} - \frac{X_o k_1}{k_1} e^{-k_2 t}$$

These two parts refer to the steady state and transient responses. Thus the first term $X_o k_1 / k_1$ corresponds to the steady state solution that is obtained at infinite time. We can see this if we take $t \to \infty$ where the second part, $(X_o k_1 / k_1) e^{-k_2 t}$ tends to zero.

When we combine both the free and forced parts of the equation we obtain the total response shown in Figure 3.

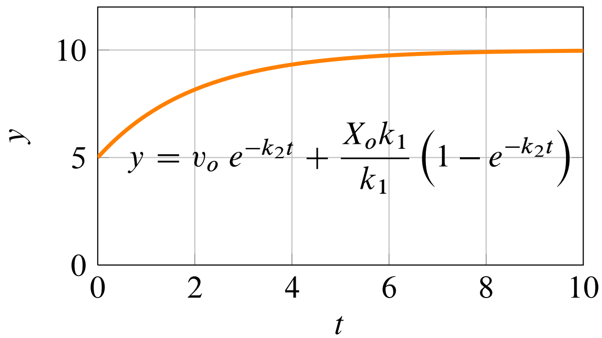

Figure 3: Total Response for a First-Order System, $X_o = 5, k_1 = 1, k_2 = 0.5$

$$y(t) = \underbrace{y(0)e^{-k_2 t}}_{\text{Free Transient}} + \underbrace{\underbrace{\frac{X_o k_1}{k_2}}_{\text{Steady State}} - \underbrace{\frac{X_o k_1}{k_2} e^{-k_2 t}}_{\text{Forced Transient}}}_{\text{Forced Response}} \tag{3}$$

Equation 3 summaries the various parts in the full response equation.

Finally, the reason why we can split the total response into the various behavioral modes is because of the **principle of superposition**.

Question 1

Consider the following two species biochemical pathway:

$$\xrightarrow{v_o} y_1 \xrightarrow{k_1\ y_1} y_2 \xrightarrow{k_2\ y_2}$$

with ODEs:

$$\frac{dy_1}{dt} = v_o - k_1 y_1$$

$$\frac{dy_2}{dt} = k_1 y_1 - k_2 y_2$$

Derive the free, forced and full Laplace **and** time-domain solutions for this system for both variables, y_1 and y_2 in response to a step function in v_o.
