Systems and Control
Week 1 and 2: Class Exercises (v1.01)

Question 1

Exercise: How you would describe each of the following systems in terms of how you might model them?

1. A game of chess
2. Two teams playing American Football
3. A forest fire
4. Gene regulation in a bacterium
5. The beating of a healthy heart
6. A tornado
7. Growth of a tumor
8. Upper atmosphere chemistry
9. A digital computer
10. Electron flow through a single transistor
11. A population of cells from which emerges a cancerous cell
12. Metabolism
13. The swimming of a single E. coli through water

Question 2

Exercise: Given the following resistor network (Figure 1), determine the voltage V_o.

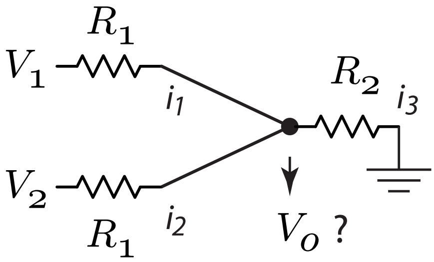

Figure 1: Resistor Network

When you've derived the relation for V_o, assume $R_2 \gg R_1$ and then infer what calculation could be done with the circuit.

Question 3

Devise a resistive circuit that can be used to compute a voltage times a constant.

Question 4

Figure 2 shows two resistor circuits. Circuit b) has a load resistor, R_3 which draws current i_3 from the potentiometer. In circuit a), $i_3 = 0$. For each circuit determine the output voltage, e_o. Explain why circuit b) would not be a good circuit for multiplying a voltage by a constant?

Question 5

For each of the following systems, decide whether the system is isolated, closed or open.

i) A system represented by a mechanical clock slowly winds down in a room controlled by a thermostat.

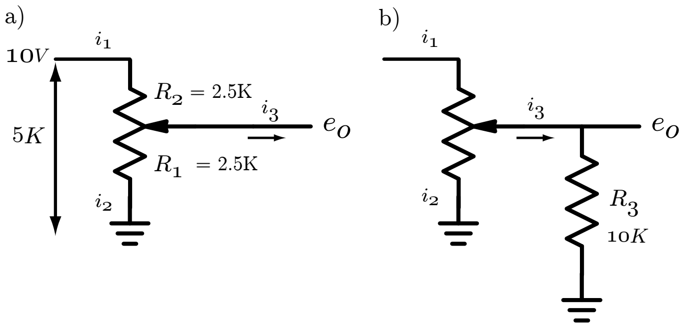

Figure 2: Effect of loading on a potentiometer circuit, a) no loading; b) with load resistor R_3

ii) A car engine running idle in the open air.

iii) A bacterial culture is grown in batch and kept in a sealed and insulated chamber.

Question 6

Discuss what a model is and write out and present to the class a succinct one or two sentence definition of a model.

Question 7

a) Identify the state variables in resistor network in Figure 1.

V_o, i_1, i_2, i_3

b) Identify the boundary parameters in the resistor network:

V_1, V_2 and ground.

c) Identify the parameters in the system:

R_1, R_2

d) Identify the state variables in the reaction pathway:

$$S_1 \xrightarrow{v_1} S_2 \xrightarrow{v_2} S_3$$

S_1 and S_3 are assumed to be free floating and not under the control of the experimenter (except at time zero when the experiment starts).

S_1, S_2, S_3

Is the reaction network open or closed?

Suggest what kinds of parameters might be found in such a model?

Question 8

Show that the function e^x is nonlinear.

Question 9

You are assigned the task of designing a clock that can be used to measure time over the period of a day, for example to indicate the passing hours. You only have at your disposal, a spring fed source of water, pipes, water tanks and a modern electric clock that you can use to calibrate your water clock. Design a clock using these tools. During its operation, the clock should not require any human intervention. The clock only has to measure the passing hours over **one** day.

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1 Class Exercises

Build and simulate the following models:

1.1 Water Tank Models

Figure 3 shows two water tanks. The first tank is fed with water at a rate dQ_1/dt. This tank drains into a second tank at a rate dQ_2/dt

which in turn drains to waste at a rate dQ_3/dt. The second tank has an addition feed of water at a rate dQ_4/dt. The height of the water level in each tank is given by Y_1 and Y_2 respectively. The resistance to outflow at each outflow pipe is given by R_1 and R_2.

Assume conservation of mass and that the rate of outflow from a tank is equal to the height of water in the tank divided by the resistance to the outflow. That is the greater the resistance the slower the outflow:

$$\frac{dQ_i}{dt} = \frac{Y_i}{R_i}$$

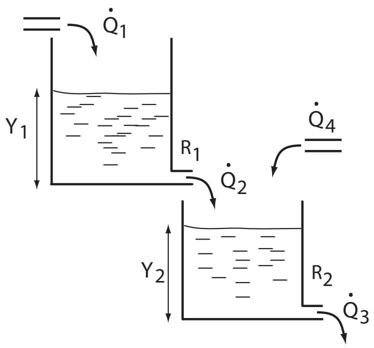

Figure 3: Water Tank Model

Questions:

a) Plot the rate of outflow as a function of the height of water for the first tank at a given resistance, R.

b) Assuming that dQ_1/dt is fixed, what do you expect to happen over time as water flows in?

c) Given your answer in b), write out the equations that predict the height of water in each tank

d) Build a computer model of the tank system and plot the filling of the tanks over time.

e) Investigate the effect of increasing and decreasing the resistance parameters, R_1 and R_2 on the model

1.2 Predator/Prey Models

a) Build a computer model of the predator/prey system described by the Lotka-Volterra system of equations:

$$\frac{dPrey}{dt} = Prey\ .\ \alpha - \beta\ .\ Prey\ .\ Predator$$

$$\frac{dPredator}{dt} = \delta\ .\ Prey\ .\ Predator - \gamma\ .\ Predator$$

where α is the growth rate of the prey, β is the rate of predation, γ is the death rate of the predator and δ is the rate of growth in the predator population.

b) Explore how the parameters affect the evolution of the populations.

1.3 Biochemical Models

Build a computer model of the following simple metabolic pathway:

$$X_0 \xrightarrow{v_1} S_1 \xrightarrow{v_2} S_2 \xrightarrow{v_3} X_1$$

Assume that the metabolites, X_o and X_1 are fixed. Assume also that v_1, v_2 and v_3 are governed by the rate laws:

$$v_1 = E_1(k_1 X_o - k_2 S_1)$$

$$v_2 = E_2(k_3 S_1 - k_4 S_2)$$

$$v_3 = E_3 \,.\, k_5 \,.\, S_2$$

where E_i is the concentration of the i^{th} enzyme and k_i are the kinetic rate constants.

a) Write out the differential equations for this system

b) Solve for the steady state

c) What is the effect of doubling all E_i by a factor α on the steady state concentrations of S_1 and S_2? Can you explain your observation?

1.4 Pharmokinetic Models

Figure 4 shows a simple compartmental model of what happens to a drug when it is injected intravenously into a patient. The model comprises of three compartments, the blood stream, the liver and the actual site of action. It is assumed that the drug can freely exchange between the liver and blood compartments but that the liver can also irreversible degrade the drug. In addition some drug is lost by excretion through the kidneys. Finally some drug accumulates irreversibly at the site of action.

We will assume that movement of the drug between compartments obeys simple first order kinetics. For example, the rate, v_2, at which drug enters the liver is given by $k_2 D_b$ where D_b is the concentration of drug in the blood stream and k_2 is the kinetic constant for the process. The complete list of rates is given below:

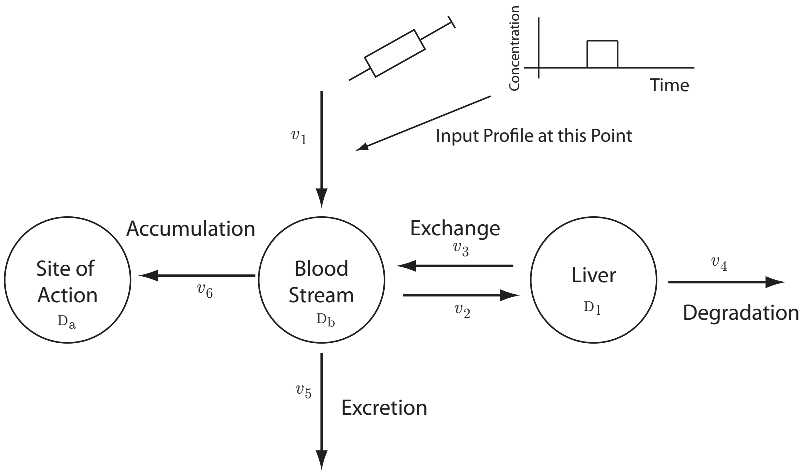

Figure 4: Simple model of a drug begin injected and distributed through the body.

$$
\begin{aligned}
v_2 &= k_2 D_b \\
v_3 &= k_3 D_l \\
v_4 &= k_4 D_l \\
v_5 &= k_5 D_b \\
v_6 &= k_6 D_b
\end{aligned}
$$

where D_b is the concentration of drug in the blood stream, D_l the concentration of drug in the liver, and D_a the concentration of drug at the site of action. Using the conservation of mass, three differential equations can be written down that describe the rate of change of drug in each of the three compartments.

Previous pharmokinetic studies of the drug indicate that the rate constants for the entry and exit of drug to and from the various compartments are given by:

$k_2 = 6.12,\ k_3 = 0.2,\ k_4 = 0.45,\ k_5 = 1,\ k_6 = 5.$

All units in seconds. Using a simulation model answer the following question.

Assume that the nurse can inject 1 mM of drug per second into the patient's blood stream. That is, in one second the concentration of drug in the blood stream increases by 1 mM (A little unrealistic perhaps). This is another way of saying that $v_1 = 1$ mM sec^{-1}. The nurse can start and stop the injection very quickly such that the profile one observes in the rate at which drug enters the blood looks like a pulse as shown in Figure 5.

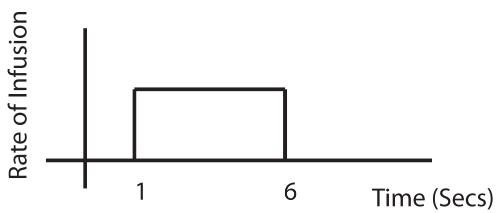

Figure 5: Drug Infusion Profile.

Assume that in order for the drug to be effective the concentration at the site of action must reach at least 2.5 mM. If the concentration of drug in the syringe is 1 mM, use the computer model to roughly estimate the minimum time the injection should last in order for the drug to be effective.