Assignment for Week 3/4

First Part: Due 4nd February (midnight on Saturday)

Second Part: Due 7th February.

Reminder: Mid Term is on the 9th of February.

Part 1

1. Define the transfer function

The **transfer function** for a given state variable is defined as the ratio of the output, $Y(s)$ to the input, $X(s)$ of the system in the Laplace domain. All initial conditions are assumed to be **zero**.

$$H(s) = \frac{Y(s)}{X(s)}$$

2. Knowing what you know about the different types of input signals, what kind of input is implied in the transfer function?

An impulse because the Laplace transform for an impulse is 1.

3. In class we discussed how the solution to a dynamical system could be spit into different responses, namely free, steady state and forced. The equation showed that the total response was a simple sum of these basic responses.

Explain **in one sentence** why the total response is a simple sum of the individual responses?

We can sum the different responses because the system in linear and thus obeys the principle of superposition.

4. A given system has the following transfer function:

$$H(s) = \frac{2s + 4}{s^2 + 5s + 6}$$

a) What is the denominator of the transfer function called?

The characteristic polynomial. If set to zero the denominator is called the characteristic equation.

b) What are the roots of the denominator called ?

Poles

c) Determine the roots of characteristic equation from the transfer function.

-3 and -2

d) What kind of dynamics would this system display if the state variable that corresponds to this transfer function was perturbed slightly away from the steady state?

Stable node

e) Roughly sketch the dynamics on a phase plot.

Sketch here

f) Carry out a simulation to show that your dynamic prediction in question (d) and (e) matches the simulation results. In order to answer this question construct a system that has the same roots. You can do this empirically by using one of the software tools that were demonstrated in class, ie find a set of values to use in the **A** matrix that gives the same roots. Build a simulation using this **A** matrix, ie solve $d\mathbf{x}/\mathbf{dt} = \mathbf{A}\mathbf{x}$.

Do simulation

g) Implement your own Newton-Raphson algorithm to find one of the roots in the characteristic equation of $H(s)$

Implement

5. In your last assignment you derived the transfer function for the following system:

$$X_0 \xrightarrow{v_1} y_1 \xrightarrow{v_2} y_2 \xrightarrow{v_3} X_1$$

Use the following rate laws:

$$v_1 = k_1 X_0 \quad v_2 = k_2 y_1 \quad v_3 = k_3 y_2$$

Use the following values $X_0 = 10; k_1 = 2.3; k_2 = 4.5; k_3 = 0.5$

Using the Newton-Raphson algorithm to determine the steady state values for y_1 and y_2. Since this system is more than one dimensional, the normal scaler version of the Newton-Raphson is insufficient. Instead the method needs to be recast as a matrix problem. Use Google to research the method for implementing the multidimensional version of the Newton-Raphson algorithm.

See Numerical Recipes by Press for detail. Implement:

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \boldsymbol{J}^{-1}\boldsymbol{F}$$

where $\boldsymbol{J}$ is the Jacobian matrix.

6. Given the following signal:

$$f(t) = 8tu(t) - 16(t - 0.5)u(t - 0.5)$$

a) Sketch the signal (manually or by software)

Rises linearly, stops at $t = 0.5$ then comes back down at the same rate. Form the same of a triangle.

b) Derive the Laplace transform for this signal, you may use the Laplace transform tables to help you.

$$f(t) = 8t\ u(t) - 16t\ u(t-0.5) + 8u(t-0.5)$$

Using the s-differentiation rule to convert $-16tu(t-0.5)$:

$$\mathcal{L} = \frac{8}{s^2} - 16e^{-0.5s}\frac{1+0.5s}{s^2} + \frac{8e^{-0.5s}}{s} = \frac{8-16e^{-0.5s}}{s^2}$$

Part 2

7. Figure 1 sketches a particular input signal, the squiggle in the middle of the plot represents a full cycle sine wave. Write out the expression you would use to represent this signal. y is the height of the signal at $t = 0$. a and b are the start and stop times for engaging and disengaging the sine wave.

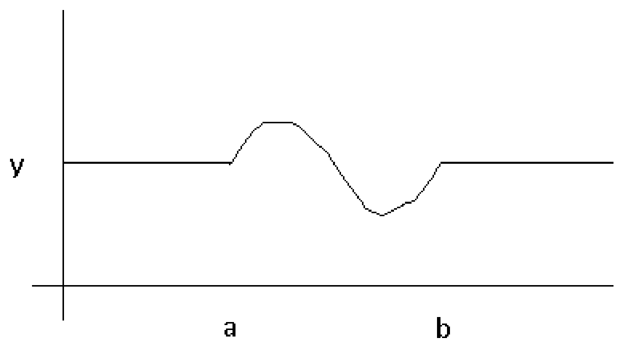

Figure 1: y is the hight of the initial step, a and b mark the start and end of the sine wave

$$yu(t) + A\sin\left((t-a)\frac{2\pi}{b-a}\right)u(t-a) - A\sin\left((t-b)\frac{2\pi}{b-a}\right)u(t-b)$$

8. Consider the simple two reaction system:

$$X_0 \xrightarrow{v_o} y_1 \xrightarrow{k\ y_1}$$

where v_o is the input rate and k the first-order rate constant. This system has the following transfer function:

$$H(s) = \frac{1}{s+k}$$

a) Apply a step function that has a height h and starts at time $= a$ and stops at time $= b$. Derive the time-domain solution.

$$x(t) = h(u(t-a) = u(t-b))$$

$$\mathcal{L}(x(t)) = h\left(\frac{1}{s}e^{-as} - \frac{1}{s}e^{-bs}\right)$$

$$Y(s) = \frac{1}{s+k}h\left(\frac{1}{s}e^{-as} - \frac{1}{s}e^{-bs}\right)$$

$$Y(s) = h\left(\frac{1}{s(s+k)}e^{-as} - \frac{1}{s(s+k)}e^{-bs}\right)$$

Inverse transform:

$$y(t) = \frac{h}{k}\left(u(t-a) - u(t-b) - u(t-a)e^{k(t-a)} + u(t-b)e^{-k(t-b)}\right)$$

b) Carry out a numerical simulation to show that the solution you derived matches the numerical simulation (Pick a suitable value for k).

Show plots

c) Sketch the signal generated by the following input equation:

$$x(t) = 2t - 2(t-12)u(t-12)$$

Ramp followed by constant value at $t = 12$

d) Apply this input to the same system, $H(s)$, and derive the time domain solution.

The Laplace transform of $x(t)$ is:

$$X(s) = \frac{2}{s^2} - \frac{2}{s^2}e^{-12s}$$

Multiply $H(s)$ into $X(s)$ to yield:

$$H(s)\ X(s) = Y(s) = \frac{2}{s^2(k+s)} - \frac{2}{s^2(k+s)}e^{-12s}$$

Partial fraction of the first term, $Y_1(s)$ yields:

$$Y_1(s) = \frac{2}{ks^2} - \frac{2}{k^2 s} + \frac{2}{k^2(k+s)}$$

Inverse Transform of $Y_1(s)$ yields:

$$y_1(t) = \frac{2}{k}tu(t) - \frac{2}{k^2}u(t) + \frac{2}{k^2}e^{-kt}$$

What about the second term? The second term is the first term shifted to the right by 12 time units (look up upshifted Laplace). That is:

$$y(t) = y_1(t) - y_1(t-12)u(t-12)$$

Inserting $y_1(t)$ into this equation yields:

$$y(t) = \frac{2}{k}tu(t) - \frac{2}{k^2}u(t) + \frac{2}{k^2}e^{-kt} - \left(\frac{2}{k}(t-12)u(t-12) - \frac{2}{k^2}u(t-12) + \frac{2}{k^2}e^{-k(t-12)}u(t-12)\right)$$

e) Plot the output y_1 and input function together to compare the input and output responses.

Show plots

9. Consider the simplest nonlinear system, a two step pathway with the first step governed by an irreversible mass-action rate law and the second step by an irreversible Michaelis-Menten rate law. The pathway has a single species, x which is governed by the nonlinear differential equation:

$$\frac{dx}{dt} = k_1 X_o - \frac{\text{Vm } x}{\text{Km} + x}$$

The steady state level for x is denoted by x_o and the parameter, Vm, has a value of Vm_o at steady state.

a) Linearize this differential equation around the steady state values, x_o and Vm_o to generate a linearized version of the differential equation.

$$\delta\dot{x} = -\left(\frac{\text{Vm Km}}{(\text{Km} + x_o)^2}\delta x + \frac{x_o}{(\text{Km} + x_o)}\delta\text{Vm}\right)$$

which describes the rate of change of perturbation in response to a step change in Vm. Note that the terms that include x_o are constant so that the equation can be reduced to

$$\delta\dot{x} = -(C_1\delta x + C_2)$$

which is a linear equation. Taking the Laplace transform on both sides yields:

$$sX(s) - \delta x_o = -\left(C_1 X(s) - \frac{C_2}{s}\right)$$

where δx_o is the initial delta perturbation in x_o. Solving for $X(s)$:

$$X(s) = \frac{\delta x_o}{s + C_1} - \frac{C_2}{s(s + C_1)}$$

Taking the inverse transform:

$$\delta x(t) = \delta x_o e^{-C_1 t} - \frac{C_2}{C_1}\left(1 - e^{C_1 t}\right)$$

The solution to this equation normal integraton is given by:

$$\delta x(t) = \left[e^{-C_1 t} - 1\right]\frac{C_2}{C_1} + \delta x_o e^{-C_1 t}$$

This equation describes the time evolution in δx as a result of a perturbation in δVm and/or δx. Note that the equation only applies to small changes in δVm (contained in C_2) and δx_o because of the linearization.

If for example we assume that the initial condition for δx to equal to zero (i.e $\delta x_o = 0$, no perturbation in x), then the steady state solution (obtained as t goes to infinity) is:

$$\delta x = -\frac{C_2}{C_1}$$

At $t = 0$ the perturbation is zero as defined by the initial condition but as t advances, $\delta x(t)$ goes negative indicating that a perturbation in Vm results in a decline in the steady state level of x. As time continues to advance, δx reaches a new steady state given by $-C_2/C_1$. Note, this is the delta change in x not the absolute value of the new level of x, which is why the negative sign makes sense here.

If on the other hand δVm is zero but δx_o is not, then $C_2 = 0$ and at steady state $\delta x = 0$, that is the system decays back to its original state.

b) What is the meaning of the $\frac{dx}{dt}$ term that is on the left-hand side of the linearized equation?

The rate of change of the disturbance

c) Take the Laplace transform of the linearized equation and derive the time dependent solution of the linearized system.

d) Does the solution you get in c) look familiar?

Yes, it represents the solution from the simple linear ode, $dx/dt = a - kx$ we saw in class.

d) Using the solution from c), confirm that if δVm is zero that the system relaxes back to the original steady state, ie $\delta x = 0$.

Confirm

e) Using the solution from c), determine the steady state change in δx when we make a change in δVm and no change to δx.

Confirm

f) Using the characteristic equation from the transfer function in c), make a statement about the stability of the system and the kind of dynamics you expect to see after a disturbance in δx.

Stable, look at roots

g) Run a simulation of the linearized and nonlinear systems. Carry out perturbation experiments in both systems by perturbing δx by differing amounts and looking at the relaxation dynamics. How well does the linear system capture the nonlinear model? With your knowledge of the Michaelis-Menten equation at what level should the species, x be relative to the enzyme's Km in order to increase the reliability of the linearized system?